Università degli Studi di Verona
Corso di Laurea Magistrale in Matematica Applicata

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Some exercises of functional analysis - A.A. 2012/13 - N.6

Pb 1. Let X and Y be normed spaces and $T, T_n : X \to Y$ linear functions. Let us set

$$A := \{x \in X : T_n x \not\to Tx\}.$$

Prove that either A is the empty set of A is dense in X.

Pb 2. Let c_{00} the space of sequences with finite support, endowed with the supremum norm. Let $(a_n) \subset \mathbb{R}$ be a sequence and set

$$\|x\|_a := \sum_{n=1}^{\infty} |a_n||x_n|, \quad x \in c_{00}.$$

Prove that $\|\cdot\|_a$ is a norm if and only if $a_n > 0$ for every $n \geq 1$. Morever, given $(a_n) \subset \mathbb{R}$ and $(b_n) \subset \mathbb{R}$ with $a_n, b_n > 0$ for every $n \geq 1$, prove that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent if and only if $0 < \inf_{n\in\mathbb{N}} \frac{|a_n|}{|b_n|} \leq \sup_{n\in\mathbb{N}} \frac{|a_n|}{|b_n|} < +\infty$.

Pb 3. Let X be a normed space and let $x, y \in X$. Prove that the function $\varphi(t) = \|x - ty\|$ attains its infimum on $\mathbb{R}$.

Pb 4. Let $T : C([0,1]) \to C([0,1])$ be a linear function such that $Tf \geq 0$ for every $f \in C([0,1])$. Prove that T is continuous and that $\|T\| = T(1)$.

Pb 5. Let X, Y be two normed spaces with $X \neq \{0\}$. Assume that $\mathcal{L}(X,Y)$ is a Banach space. Prove that, in turn, Y is a Banach space.

Pb 6. Let $1 \leq p < \infty$. Find an isometry $T : \ell^p \to \mathcal{L}(L^p([0,1]), L^p([0,1]))$.

Pb 7. Let $1 \leq p < \infty$ and consider $T : \ell^p(\mathbb{N}) \to L^p(\mathbb{R}^+)$ defined by

$$T(x)(y) := \sum_{n=1}^{\infty} x_n \chi_{[n-1,n]}(y), \quad y \in \mathbb{R}^+, \ x \in \ell^p(\mathbb{N}).$$

Prove that T is an isometry.

Pb 8. Let $Y = \{P \in L^2([0,1]) : P \text{ is a polynomial}, P(1) = 0\}$. Prove that Y is dense in L^2.

Pb 9. Let $\varphi : c_0 \to \mathbb{R}$ defined by

$$\varphi(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^{n-1}}, \quad x \in c_0.$$

Prove that $\varphi \in (c_0)'$ with $\|\varphi\| = 2$ and that φ does not achieve its norm on the unit disk of c_0.

Pb 10. Prove that the set of functions in $C^1([0,1])$ such that

$$|f(0)| \leq C_1, \quad \int_0^1 |f'(\sigma)|^2 d\sigma \leq C_2$$

for some positive constants C_1, C_2 is relatively compact in $C([0,1])$.

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