MATEMATICA
Università di Verona
Laurea in Biotecnologie - A.A. 2012/13
Lezione di venerdì 09/11/2012

Limiti in forma indeterminata

$$\lim_{x\to+\infty} \frac{a^x}{x^n} = +\infty \quad \forall n>0,\ a>1 \qquad \lim_{x\to+\infty} \frac{\log_a x}{x^\alpha} = 0 \quad \forall \alpha>0,\ a>1$$

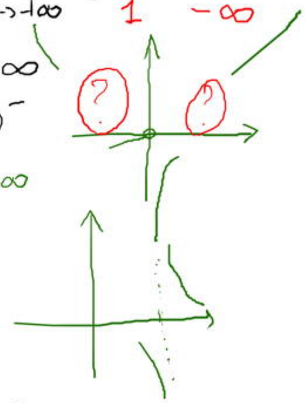
$$\lim_{x\to 0^+} x \log x = 0^- \qquad \lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2} \qquad \lim_{x\to 0} \frac{a^x-1}{x} = \log a$$

$$\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \frac{1}{\log a} \qquad \lim_{x\to 0} \frac{(1+x)^\alpha - 1}{x} = \alpha$$

$$\lim_{x\to 1} \sin(\log x) = 0 \qquad \lim_{x\to+\infty} \left(a^{1/x} - x\right) = -\infty \quad 1 \quad -\infty \qquad \lim_{x\to 0} \frac{e^{\frac{1}{x-2}}}{\sqrt{x+4}} \quad \frac{1/\sqrt{e}}{2} = \frac{1}{2\sqrt{e}}$$

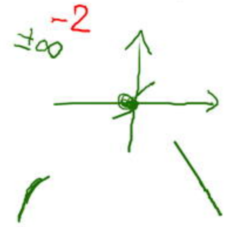
$$\lim_{x\to 0^\pm} (3x^2-5x-6)\, e^{1/x} \;\; {}^{\pm\infty} = \begin{matrix} -\infty \\ 0^- \end{matrix}$$

$+\infty$, $-\infty$; -6; $+\infty$; 0^+; $+\infty$; 1; $+\infty$



$$\lim_{x\to+\infty} \frac{\log(x^2+4)}{1/x - 2} = -2\log 2 - \infty$$

$+\infty$; -2; $\pm\infty$; $0^\pm$; $-\infty$

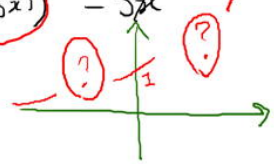


$$\lim_{x\to 1^\pm} \frac{\cos^2(\pi x)}{\log x} = \pm\infty \qquad 1, \; 0^\pm$$

$$\lim_{x\to-\infty} \frac{e^{x-x^2} + 2\arctan x}{\log(x^4-1)} \qquad 0^+ \; (-\infty), \; -\pi, \; -\pi, \; +\infty \qquad 0^-$$

$$\lim_{x\to 0} \frac{\sin 3x}{\log(1-5x)} = \lim_{x\to 0} \frac{\sin(3x)}{3x}\, 3\cancel{x}\, \frac{-5x}{\log(1+(-5x))} \cdot \frac{1}{-5\cancel{x}} = -3/5 \qquad 1, \quad \frac{1}{1} = 1$$

$$\lim_{x\to\mp\infty} \frac{e^x - 1}{x} = \begin{matrix} 0^+ \\ \end{matrix} \qquad = \lim_{x\to+\infty} \frac{e^x}{x}\left(1 - \frac{1}{e^x}\right) = +\infty \qquad +\infty, \; 0$$



$$\lim_{x\to+\infty} \frac{e^x - 3x - \cos x}{4e^x + e^{-x}} = \lim_{x\to+\infty} \frac{\cancel{e^x}\left(1 - \frac{3x}{e^x} - \frac{\cos x}{e^x}\right)}{\cancel{e^x}\left(4 + \frac{e^{-x}}{e^x}\right)} = \frac{1}{4}$$

$$\lim_{x\to-\infty} \frac{e^x - 3x - \cos x}{4e^x + e^{-x}} = \lim_{x\to-\infty} \frac{x\left(\frac{e^x}{x} - 3 - \frac{\cos x}{x}\right)}{e^{-x}\left(\frac{4e^x}{e^{-x}} + 1\right)} = \lim_{t\to+\infty} \frac{3t}{e^t} = 0 \qquad t = -x$$

$$\lim_{x\to-\infty} \frac{x^3 - 2x^2 + x + 5}{3x^n + 1} \quad (n \in \mathbb{N}) \qquad t = -x \qquad \lim_{t\to+\infty} \frac{-t^3 - 2t^2 - t + 5}{3(-t)^n + 1}$$

$$= \lim_{t\to+\infty} \frac{t^3\left(-1 - \frac{2}{t} - \frac{1}{t^2} + \frac{5}{t^3}\right)}{t^n\left(3(-1)^n + \frac{1}{t^n}\right)}$$

se $n \le 2$: $+\infty$ $(n=1)$, $-\infty$ $(n=2)$ $\quad (-1)^n t^n$

se $n = 3$: $1/3$ $(n=3)$

se $n \ge 4$: 0

$$\lim_{x\to 3^+} \frac{\log(2x-5)}{(x-3)^\alpha} \quad (\alpha \in \mathbb{R}) \qquad 0$$

se $\alpha > 0$: 0 ~~~~
se $\alpha = 0$: 1
se $\alpha < 0$: $+\infty$

se $\alpha \le 0$: $\lim = 0$

se $\alpha > 0$: f.i.!!

$x - 3 = t$

$x = t + 3$

$$\lim_{t\to 0^+} \frac{\log(1+2t)}{t^\alpha} = \lim_{t\to 0^+} \frac{\log(1+2t)}{2t} \cdot \frac{2t}{t^\alpha} = \qquad 2t^{1-\alpha} \quad (\alpha < 1)$$

1

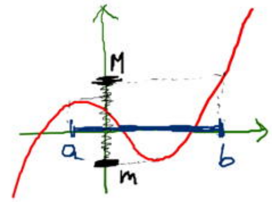
se $1-\alpha > 0$: 0

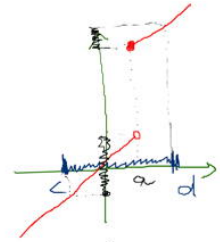
se $1-\alpha = 0$ $(\alpha = 1)$: 2

se $1-\alpha < 0$ $(\alpha > 1)$: $+\infty$

FUNZIONI CONTINUE: PROPRIETA' ULTERIORI

- Una f.c. manda intervalli in intervalli
- Una f.c. manda intervalli chiusi e limitati (COMPATTI) in intervalli chiusi e limitati (COMPATTI)

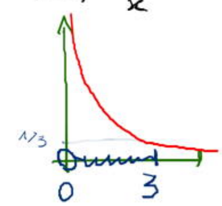




WEIERSTRASS

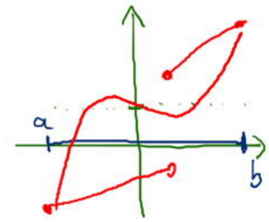
UNA F.C. SU UN DOMINIO CHIUSO E LIMITATO AMMETTE SEMPRE MAX E MIN ASSOLUTI

$f(x) = \frac{1}{x}$



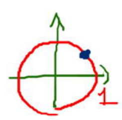
T. DEGLI ZERI

Se $f : [a,b] \to \mathbb{R}$ è continua e $f(a) < 0$, $f(b) > 0$ allora $\exists\, x \in\,]a,b[\, : f(x) = 0$

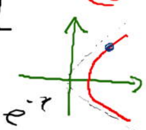


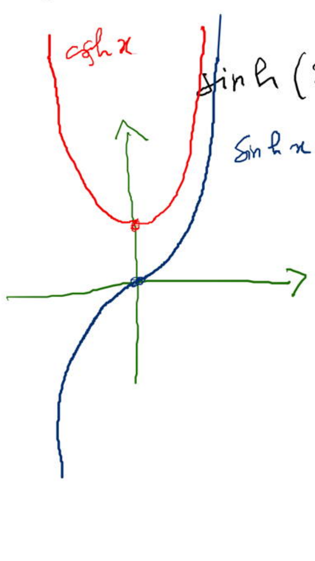
FUNZIONI IPERBOLICHE

$\cosh : \mathbb{R} \to \mathbb{R} \qquad \cosh(x) := \dfrac{e^x + e^{-x}}{2} \qquad \cos^2 x + \sin^2 x = 1$



$\sinh : \mathbb{R} \to \mathbb{R} \qquad \sinh(x) := \dfrac{e^x - e^{-x}}{2} \qquad \cosh^2 x - \sinh^2 x = 1$





$$\sinh(2x) = \frac{e^{2x} - e^{-2x}}{2} = \frac{(e^x + e^{-x})(e^x - e^{-x})}{2} = 2\,\frac{e^x + e^{-x}}{2}\,\frac{e^x - e^{-x}}{2}$$

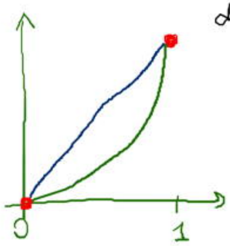
$$= 2 \sinh x \cosh x$$

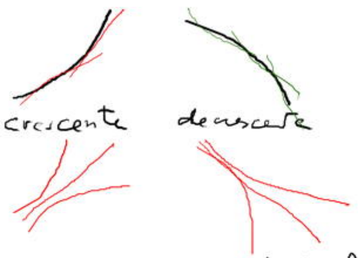
DERIVAZIONE : studio della RAPIDITÀ di VARIAZIONE di una funzione. "LA PENDENZA DEL GRAFICO"

$f, g : [0,1] \to \mathbb{R}$

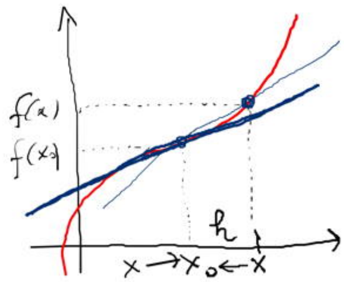
$f(x) = x$

$g(x) = x^2$



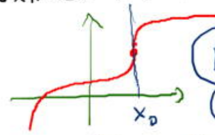


RAPP. INCREMENTALE

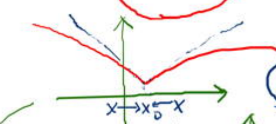


Se $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ esiste finito si dice che f è DERIVABILE in x_0 e il valore $f'(x_0)$

DERIVATA DI f IN x_0

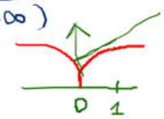


NO ($+\infty$)

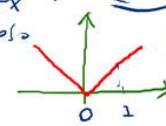


NO

ANGOLOSO



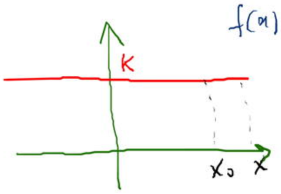
$f(x) = \sqrt{|x|}$



$g(x) = |x|$

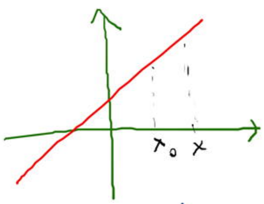
$h = x - x_0$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$



$f(x) = K \quad \forall x$

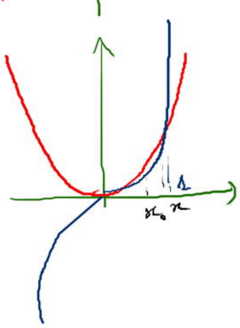
$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{K - K}{x - x_0} = 0 \qquad f'(x) \equiv 0$$



$f(x) = mx + q$

$f'(x) = m \quad \forall x$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{mx + \cancel{q} - (mx_0 + \cancel{q})}{x - x_0} = m$$



$f(x) = x^2$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \lim_{x \to x_0} \frac{\cancel{x^2 - x_0^2}}{\cancel{x - x_0}} = 2x_0 \qquad x + x_0$$

$x^2 + x x_0 + x_0^2$

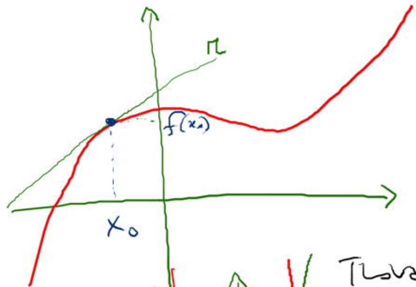
$f'(x_0) = 2x_0$

$g(x) = x^3$

$$\lim_{x \to x_0} \frac{\cancel{x^3 - x_0^3}}{\cancel{x - x_0}} = 3x_0^2$$

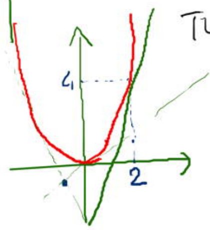
$f(x) = x^n$

$$\boxed{f'(x_0) = n x^{n-1}}$$



RETTA TANGENTE

$$\boxed{y - f(x_0) = f'(x_0)(x - x_0)}$$

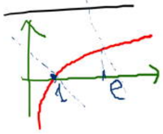


Trovare la retta tangente alla parabola $y = x^2$ sopra $x_0 = 2$

$f(x) = x^2 \qquad f'(x_0) = 2x_0 \qquad f'(2) = 4$

$y - 4 = 4(x - 2) \qquad y = 4x - 4$

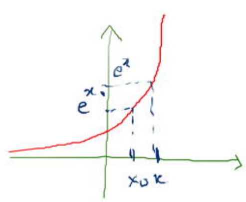
Esercizi



(1) Quali rette tg. al grafico di $f(x) = x^2$ passano per $(-1, -1)$?

(2) Trovare la retta ortogonale al grafico di $\log x$ sopra $x_0 = 1, x_0 = e$

- $f(x) = e^x$



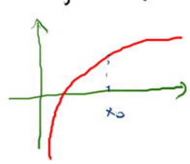
$$\lim_{x \to x_0} \frac{e^x - e^{x_0}}{x - x_0} = \lim_{h \to 0} \frac{e^{x_0+h} - e^{x_0}}{h}$$

$$= \lim_{h \to 0} \frac{e^{x_0} e^h - e^{x_0}}{h} = \lim_{h \to 0} e^{x_0} \left(\underbrace{\frac{e^h - 1}{h}}_{1} \right) = e^{x_0}$$

$f(x) = a^x \qquad f'(x_0) = a^x \cdot (\log a)$

- $f(x) = \log x$

$f'(x_0) = \dfrac{1}{x_0}$



$$\lim_{x \to x_0} \frac{\log(x) - \log(x_0)}{x - x_0} = \lim_{h \to 0} \frac{\log(x_0 + h) - \log(x_0)}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{\log(1 + h/x_0)}{h/x_0}}_{1} \frac{1}{x_0} = \frac{1}{x_0}$$

$f(x) = \log_a x \qquad \forall a > 0, \; a \neq 1$

$f'(x_0) = \dfrac{1}{x_0 \log a}$

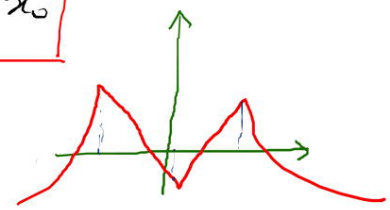
$A' \subseteq A$

$f : A \to \mathbb{R} \quad \rightsquigarrow \quad f' : A' \longrightarrow \mathbb{R}$ FUNZIONE DERIVATA

ES $\quad f(x) = x^2 \qquad f'(x) = 2x$

$$\boxed{f \text{ derivabile in } x_0 \implies f \text{ continua in } x_0}$$

$$\lim_{x \to x_0} \left(f(x) - f(x_0) \right) = \lim_{x \to x_0} \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{f'(x_0)} (x - x_0) = 0$$



$$\boxed{\begin{array}{ll} f, g \text{ derivabili in } x_0 & \implies \quad \alpha f(x) + \beta g(x) \quad \text{derivabile in } x_0 \\ \alpha, \beta \in \mathbb{R} & \quad\quad \left(\alpha f(x) + \beta g(x) \right)' = \alpha f'(x) + \beta g'(x) \end{array}}$$

$$\implies \quad f(x)\, g(x) \quad \text{pure}$$

$$\cancel{\left(f(x)\, g(x) \right)' = f'(x)\, g'(x)}$$

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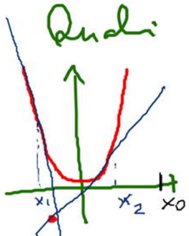
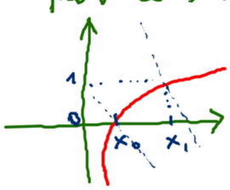
Quali rette tangenti al grafico di $f(x) = x^2$ passano per $(-1,-1)$?



$x_0 \in \mathbb{R}$ $\quad y - f(x_0) = f'(x_0)(x - x_0)$ $\quad y - x_0^2 = 2x_0(x - x_0)$ $\quad y = 2x_0 x - x_0^2$

$-1 = 2x_0(-1) - x_0^2$ $\quad x_0^2 + 2x_0 - 1 = 0$ $\quad x_{1,2} = -1 \pm \sqrt{2}$

Trovare le rette $\perp$ al grafico di $\log x$ sopra i punti $x_0 = 1$ e $x_1 = e$



$$y - \log x_0 = -\frac{1}{1/x_0}(x - x_0) \qquad y = -x_0 x + x_0^2 + \log x_0$$

$x_0 = 1$: $\quad y = -x + 1$

$x_1 = e$: $\quad y = -ex + e^2 + 1$

$f(x), g(x)$ siano derivabili in x_0 $\Longrightarrow$ anche $f(x)g(x)$ lo è, e vale

$$\boxed{\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x)\Big|_{x=x_0}}$$

REGOLA DI LEIBNIZ

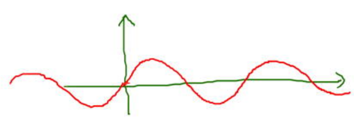
$$\big(f_1(x) f_2(x) \cdots f_n(x)\big)' = f_1'(x) f_2(x) \cdots f_n(x) + \cdots + f_1(x) f_2(x) \cdots f_n'(x)$$

Ex $\quad h(x) = 5x^5 e^x + 1 \qquad h'(x) = 5\,(x^5 e^x)' + 0 = 5(5x^4 e^x + x^5 e^x)$

$$\boxed{\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \qquad \text{in part.: } \left(\frac{1}{g(x)}\right)' = -\frac{g'(x)}{g(x)^2}}$$

Ex $\quad K(x) = \dfrac{3x^3 + \log x}{3e^x + x} \qquad K'(x) = \dfrac{\left(3\cdot 3x^2 + \frac{1}{x}\right)(3e^x + x) - (3x^3 + \log x)(3e^x + 1)}{(3e^x + x)^2}$

$f(x) = \sin x$



PROSTAFERESI !!

$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{h\to 0} \frac{\sin(x_0+h) - \sin x_0}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{2}\cos\left(x_0 + h/2\right) \sin\left(h/2\right)}{\cancel{h}/2} = \cos(x_0)$$

$$\frac{\sin(h/2)}{h/2} \to 1$$

$\sin'(x) = \cos x$

$\cos'(x) = -\sin(x)$

$tg(x) = \dfrac{\sin x}{\cos x} \qquad tg'(x) = \dfrac{\sin'(x)\cos x - \sin x \cdot \cos'(x)}{\cos^2 x} = \dfrac{1}{\cos^2 x} \qquad x \neq \pi/2 + k\pi$

$cotg(x) = \dfrac{\cos x}{\sin x} = \dfrac{1}{tg\, x} \qquad cotg'(x) = -\dfrac{1}{\sin^2 x} \qquad x \neq k\pi \qquad \forall k \in \mathbb{Z}$

$$f(x) = e^{3x} \rightsquigarrow f'(x) = ???$$

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\alpha(x)=3x"] \arrow[rr, bend right, "f(x)"'] & \mathbb{R} \arrow[r, "\beta(x)=e^x"] & \mathbb{R} \end{tikzcd}$$

$$f = \beta \circ \alpha \qquad f(x) = \beta(\alpha(x))$$

$$\boxed{(\beta\circ\alpha)'(x) = \beta'(\alpha(x))\cdot\alpha'(x)}$$

REGOLA DELLA CATENA

Ex. $f'(x) = e^{3x}\cdot 3 = 3e^{3x}$

$g(x) = \sin(\log x) \qquad g'(x) = \cos(\log x)\cdot\frac{1}{x}$

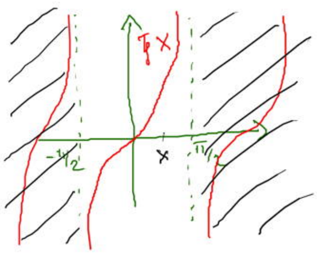
$h(x) = \log(\sin x) \qquad h'(x) = \frac{1}{\sin x}\cdot(\cos x) = \cot x$

E se ci fosse $\ell(x) = \sin(\log^3(2x+5))$

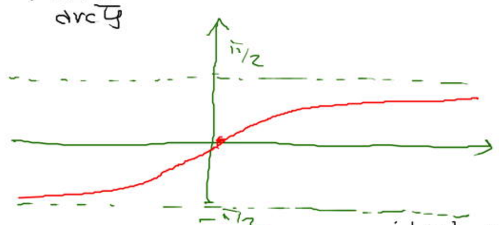
$$\mathbb{R} \xrightarrow{\alpha(x)=2x+5} \mathbb{R} \xrightarrow{\beta(x)=\log x} \mathbb{R} \xrightarrow{\gamma(x)=x^3} \mathbb{R} \xrightarrow{\delta(x)=\sin x} \mathbb{R}$$

$$\ell'(x) = (\delta\circ\gamma\circ\beta\circ\alpha)'(x) = \delta'(\gamma(\beta(\alpha(x))))\cdot\gamma'(\beta(\alpha(x)))\cdot\beta'(\alpha(x))\cdot\alpha'(x)$$

$$= \cos(\log^3(2x+5))\cdot 3\log^2(2x+5)\cdot\frac{1}{2x+5}\cdot 2 = \frac{6}{2x+5}\log^2(2x+5)\cos(\cdots)$$



$$\left]-\tfrac{\pi}{2},\tfrac{\pi}{2}\right[\underset{\text{arctg}}{\overset{\text{tg}}{\rightleftarrows}} \mathbb{R} \qquad \text{biiettiva}$$



$\text{arctg}'(x) = ?$

$\text{arctg}(\text{tg}\, x) = x \qquad \alpha = \text{tg}, \ \beta = \text{arctg} \qquad id(x) = x \qquad \beta\circ\alpha = id \qquad (\beta\circ\alpha)' = id' = 1$

$\beta'(\alpha(x))\,\alpha'(x) = 1 \qquad \beta'(\alpha(x)) = \frac{1}{\alpha'(x)} = \frac{1}{1/\cos^2 x} = \cos^2 x$

$t = \text{tg}\, x$

$x = \text{arctg}\, t$

$$\beta'(t) = \cos^2(\overset{x}{\text{arctg}\, t}) \overset{\text{TRG}}{=} \frac{1}{1+t^2}$$

$$\boxed{\cos\alpha = \frac{1}{\sqrt{1+\text{tg}^2\alpha}}}$$

$$\boxed{\text{arctg}'(x) = \frac{1}{1+x^2}}$$

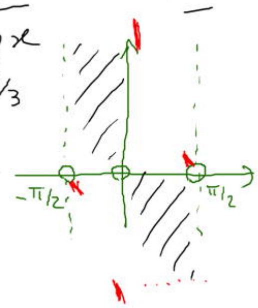
$$\boxed{\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}}$$

$f(x) = \dfrac{1}{\sqrt[3]{\operatorname{tg} x}}$ Dom. $\operatorname{tg} x \neq 0 \quad x \neq k\pi$
$x \neq \frac{\pi}{2} + k\pi$ $x \neq k\frac{\pi}{2}$

Funzione: periodica di periodo π
dispari $f(-x) = -f(x)$

$f(x) = (\operatorname{tg} x)^{-1/3}$

$\mathbb{R} \xrightarrow{\operatorname{tg} x} \mathbb{R} \xrightarrow{x^{-1/3}} \mathbb{R}$



$\lim_{x\to 0^+} f(x) = +\infty \qquad \lim_{x\to \frac{\pi}{2}^-} f(x) = 0^+$

$f'(x) = -\frac{1}{3}(\operatorname{tg} x)^{-\frac{1}{3}-1} \cdot \frac{1}{\cos^2 x} = -\dfrac{1}{3\operatorname{tg} x \sqrt[3]{\operatorname{tg} x} \cdot \cos^2 x}$

$g(x) = \log^3(x\sin x - 1) \qquad g'(x) = 3\log^2(x\sin x - 1) \cdot \frac{1}{x\sin x - 1} \cdot (1\sin x + x\cos x)$

$h(x) = e^{\cos x} \qquad h'(x) = e^{\cos x}(-\sin x) = -\sin x\, e^{\cos x}$

$\ell(x) = \cos(e^x) \qquad \ell'(x) = -\sin(e^x) \cdot e^x = -e^x \sin(e^x)$

$\alpha(x) = \sinh(x) = \dfrac{e^x - e^{-x}}{2} \qquad \alpha'(x) = \frac{1}{2}(e^x - e^{-x}(-1)) = \cosh(x)$

$\sinh'(x) = \cosh(x)$ $\cosh'(x) = \sinh(x)$

$f(x) = \sin^7(3x-5) \qquad f'(x) = 7\sin^6(3x-5) \cdot \cos(3x-5) \cdot 3 = 21\sin^6(3x-5)\cos(3x-5)$

$g(x) = e^{\frac{x}{x-1}} + 8x\cos 2x \qquad g'(x) = e^{\frac{x}{x-1}} \cdot \dfrac{1\cdot(\cancel{x}-1) - \cancel{x}\cdot 1}{(x-1)^2} + 8\left(1\cdot\cos 2x + x(-\sin 2x)\cdot 2\right)$

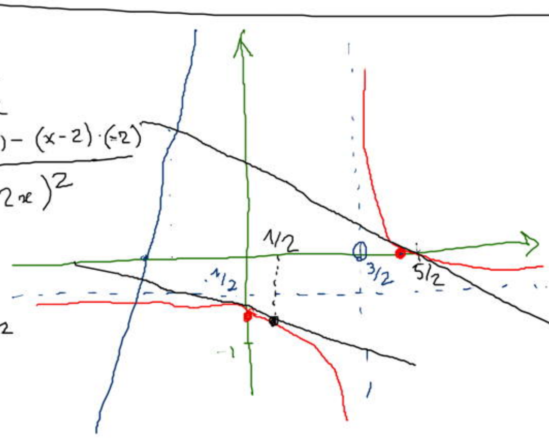
$= -\dfrac{e^{x/x-1}}{(x-1)^2} + 8(\cos 2x - 2x\sin 2x)$

$f(x) = \dfrac{x-2}{3-2x}$

$f'(x) = \dfrac{1\cdot(3-2x) - (x-2)\cdot(-2)}{(3-2x)^2}$

$= \dfrac{3 - \cancel{2x} + \cancel{2x} - 4}{(3-2x)^2}$

$= -\dfrac{1}{(3-2x)^2}$



Quali rette tangenti al grafico di $f(x)$ sono $\perp$ retta $4x - y + 5 = 0$?
$y = 4x + 5$

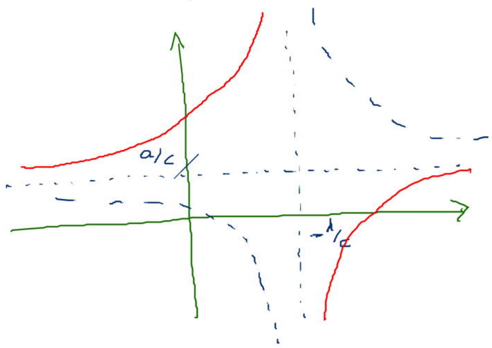
$x_0 \neq 3/2$

Pendenza del graf. di f sopra x_0: $f'(x_0)$

$f'(x_0) = -\dfrac{1}{(3-2x_0)^2} = -\dfrac{1}{4}$

$(3-2x_0)^2 = 4 \qquad 3 - 2x_0 = \pm 2$ $x_0 = 1/2$, $x_0 = 5/2$

FUNZIONI OMOGRAFICHE



$$f(x) = \frac{ax+b}{cx+d} \qquad \boxed{\begin{array}{c} c \neq 0 \\ \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0 \end{array}}$$

$x \neq -d/c$

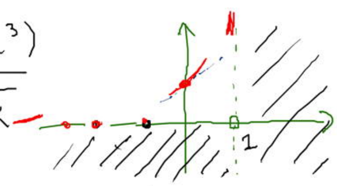
$f(x) = 0 \qquad x = -b/a$

$f(0) = b/d \quad (\text{se } d \neq 0)$

$$\lim_{x\to\pm\infty} f(x) = a/c$$

$$\lim_{x\to -d/c} f(x) = \infty$$

$$h(x) = \frac{\cos^2(x^3)}{\sqrt{1-x}}$$



$h(x) = 0 \qquad \cos^2(x^3) = 0 \qquad \cos(x^3) = 0$

$x^3 = \pi/2 + k\pi \qquad k \in \mathbb{Z}$

$x = \sqrt[3]{\pi/2 + k\pi} \qquad \underline{k \leq -1}$

$$\lim_{x\to-\infty} h(x) = 0^+ \qquad \lim_{x\to 1^-} h(x) \left(\frac{\cos^2(1)}{0^+}\right) = +\infty \qquad h(0) = 1$$

$$h'(x) = \frac{\left(2\cos(x^3)\cdot(-\sin(x^3))\cdot 3x^2\right)\sqrt{1-x} - \cos^2(x^3)\cdot\frac{1}{2}(1-x)^{1/2-1}\cdot(-1)}{1-x}$$

$$= \frac{-6x^2\cos(x^3)\sin(x^3)\sqrt{1-x} + \dfrac{\cos^2(x^3)}{2\sqrt{1-x}}}{1-x}$$

$$= \frac{-12x^2\cos(x^3)\sin(x^3)(1-x) + \cos^2(x^3)}{2(1-x)\sqrt{1-x}} \qquad h'(0) = \frac{1}{2}$$

REGOLA DI DE L'HÔPITAL

$$\lim_{x\to x_0} \frac{f(x)}{g(x)}$$

x_0: può essere anche $\pm\infty$

Supponiamo che:

(1) Si sia in una forma indeterminata $\frac{0}{0}$ oppure $\frac{\infty}{\infty}$

(2) esista $\lim_{x\to x_0} \frac{f'(x)}{g'(x)}$ in $\tilde{\mathbb{R}}$

In tal caso $\boxed{\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}}$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1 \qquad \lim_{x\to 0} \frac{\log(1+x)}{x} = \lim_{x\to 0} \frac{\frac{1}{1+x}\cdot 1}{1} = 1$$

$$\lim_{x\to +\infty} e^x/x = \lim_{x\to +\infty} e^x/1 = +\infty \qquad \lim_{x\to 0} \frac{x-\sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x}$$

$$\lim_{x\to +\infty} e^x/x^3 = \lim_{x\to +\infty} e^x/3x^2 = \lim_{x\to +\infty} e^x/6x = \lim_{x\to +\infty} e^x/6 = +\infty \qquad \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$$

Alcuni esempi di uso "improprio" di de l'Hôpital

- $\lim_{x\to 1^+} \frac{x}{\log x} \; \frac{1^+}{0^+} = +\infty$ Forma determinata $\qquad \lim_{x\to 1^+} \frac{1}{1/x} = 1$???

- $f(x) = \frac{3x+\sin x}{\cos 2x - 4x} \; \frac{\infty}{\infty} \qquad \lim_{x\to+\infty} f(x) = ? \qquad \lim_{x\to+\infty} \frac{3+\cos x}{-2\sin 2x - 4} \;\nexists$

$$= \lim_{x\to+\infty} \frac{3\cancel{x}\left(1+\underbrace{\frac{\sin x}{3x}}_{0}\right)}{-4\cancel{x}\left(\underbrace{-\frac{\cos 2x}{4x}}_{0}+1\right)} = -3/4$$

1

- $\lim_{x\to 0^\pm} \frac{1}{x} e^{-1/x}$ ($\frac{1}{x}$: $\pm\infty$)

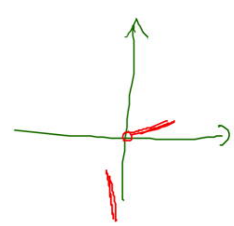
$$\lim_{x\to 0^-} \underbrace{\frac{1}{x}}_{-\infty} \underbrace{e^{-1/x}}_{+\infty} = -\infty$$

$$\lim_{x\to 0^+} \underbrace{\frac{1}{x}}_{+\infty} \underbrace{e^{-1/x}}_{0^+}$$

$$\| $$

$$\lim_{x\to 0^+} \frac{1/x}{e^{1/x}}$$

$$\lim_{x\to 0^+} \frac{\cancel{-1/x^2}}{\cancel{-1/x^2}\, e^{1/x}} = 0^+$$



Esercizi

$$\lim_{x\to 0} \frac{\log(1+x)-x}{3x^2+x^3} \qquad \lim_{x\to 1} \frac{\log x + 1 - \sqrt{2x+1}}{(x-1)^n} \quad n\in\mathbb{N}$$

$$\lim_{x\to -\infty} x\left(\pi + 2\operatorname{arctg} x\right) \qquad \lim_{x\to +\infty} \frac{e^{2x}+x^3+\cos x}{3e^{2x}-5}$$

$$\lim_{x\to 0} \frac{2\sin(2x)+\log(1-3x)-x}{x^n} \quad (n\in\mathbb{N})$$

MATEMATICA
Università di Verona
Laurea in Biotecnologie - A.A. 2012/13
Lezione di martedì 20/11/2012

- $$\lim_{x\to 0}\frac{\log(1+x)-x}{3x^2+x^3}\overset{H}{=}\lim_{x\to 0}\frac{\frac{1}{1+x}-1}{6x+3x^2}\overset{H}{=}\lim_{x\to 0}\frac{-\frac{1}{(1+x)^2}}{6+6x}=-\frac{1}{6}$$

$$\lim_{x\to 1}\frac{\log x+1-\sqrt{2x-1}}{(x-1)^n}\quad (n\in\mathbb{N})\qquad \lim_{x\to 1}\frac{\frac{1}{x}-\frac{1}{\cancel{2}}(2x-1)^{-1/2}\cdot\cancel{2}}{n(x-1)^{n-1}}=\lim_{x\to 1}\frac{\frac{1}{x}-\frac{1}{\sqrt{2x-1}}}{n(x-1)^{n-1}}$$

$-(2x-1)^{-3/2}$

SE $n=1$: $\lim=0$

invece se $n\geq 2$: $\frac{0}{0}$!

$$\lim_{x\to 1}\frac{-\frac{1}{x^2}-\left(-\frac{1}{\cancel{2}}\right)(2x-1)^{-3/2}\cdot\cancel{2}}{n(n-1)(x-1)^{n-2}}=\lim_{x\to 1}\frac{-\frac{1}{x^2}+\frac{1}{(2x-1)^{3/2}}}{n(n-1)(x-1)^{n-2}}$$

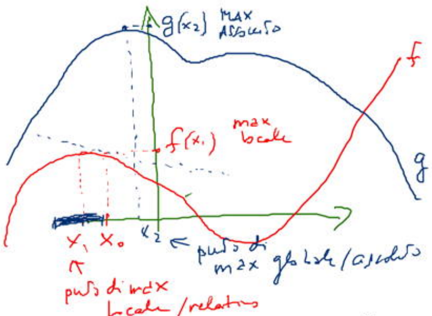
se $n=2$: $\lim=0$

invece se $n\geq 3$: $\frac{0}{0}$!

$$\lim_{x\to 1}\frac{-(-2)x^{-3}-\frac{3}{\cancel{2}}(2x-1)^{-5/2}\cdot\cancel{2}}{n(n-1)(n-2)(x-1)^{n-3}}=\lim_{x\to 1}\frac{\frac{2}{x^3}-\frac{3}{(2x-1)^{5/2}}}{n(n-1)(n-2)(x-1)^{n-3}}=-\frac{1}{n(n-1)(n-2)}$$

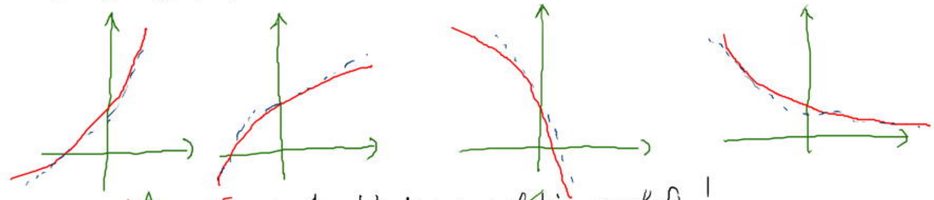
- $$\lim_{x\to -\infty}x\,(\pi+2\operatorname{arctg}x)=\lim_{x\to-\infty}\frac{\pi+2\operatorname{arctg}x}{1/x}\overset{H}{=}\lim_{x\to-\infty}\frac{2\cdot\frac{1}{1+x^2}}{-\frac{1}{x^2}}=\lim_{x\to-\infty}\left(-2\,\frac{x^2}{x^2+1}\right)=-2$$

DERIVATA PRIMA E CRESCENZA / DECRESCENZA DI UNA FUNZIONE

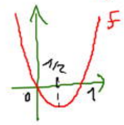


CRESCENTE: $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$

DECRESCENTE: $x_1 \le x_2 \Rightarrow f(x_1) \ge f(x_2)$



Es. $f(x) = x^2 - x$



$x = 1/2$ pto di min ~~relativo~~ assoluto!

$f(1/2) = -1/4$ min ~~relativo~~ assoluto!

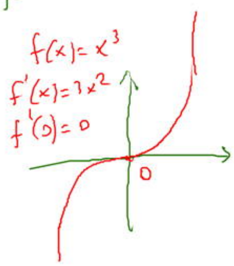
x_0 è un ESTREMANTE (REL/ASS) per f se è o pto di min rel/ass o pto di max rel/ass per f

$f(x_0)$ è un ESTREMO (REL/ASS) per f

$f(x) = x^3$

$f'(x) = 3x^2$

$f'(0) = 0$



Prop. Sia A un intervallo aperto di $\mathbb{R}$, $f : A \to \mathbb{R}$ derivabile

(a) Sia $c \in A$. Allora c estremante loc. per $f \Rightarrow f'(c) = 0$ (c è pto stazionario per f)

(b) $f' \equiv 0$ ($f'(x) = 0\ \forall x \in A$) $\Longleftrightarrow$ f è costante in A

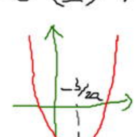
(c) f crescente in A $\Longleftrightarrow$ $f'(x) \ge 0\ \forall x \in A$
(cioè se $x_1, x_2 \in A$ $x_1 \le x_2 \Rightarrow f(x_1) \le f(x_2)$)

(d) f strettamente crescente in A $\Longleftarrow$ $f'(x) > 0\ \forall x \in A$
(cioè se $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$) ← Ex $f(x) = x^3$ $f'(0) = 0$

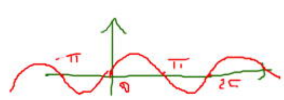
Ex $f(x) = ax^2 + bx + c$ $a > 0$ $f'(x) = 2ax + b$ $f'(x) = 0 \Longleftrightarrow x = -b/2a$

$f'(x) > 0$ $2ax + b > 0$ $x > -b/2a$

	$-b/2a$	
f'	$-$	$+$
f	↘	↗
	MIN ASS	

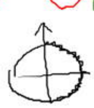


Ex $g(x) = \sin x$



$g'(x) = \cos x$ $g'(x) = 0$ $\cos x = 0$ $x = \pi/2 + k\pi$

$g'(x) > 0$
$\cos x > 0$



$-\pi/2 + 2k\pi < x < \pi/2 + 2k\pi$

	$-\pi$ … $-\pi/2$	$-\pi/2$ … 0 … $\pi/2$	$\pi/2$ … π
g'	$-$	$+$	$-$
g	↘	↗	↘
	MIN		MAX

Ex. $f(x) = \dfrac{ax+b}{cx+d}$ $f'(x) = \dfrac{a(cx+d) - (ax+b)c}{(cx+d)^2} = \dfrac{ad-bc}{(cx+d)^2}$

$f'(x) = 0$ MAI $f'(x)$ ha segno costante (f è sempre crescente o sempre decrescente, strettamente!)

$f(x) = \dfrac{x^3+26}{x^2-1}$

Dominio $x \neq \pm 1$ No simmetrie, no periodo

Zeri: $f(x) = 0 \iff x^3 = -26$ $x = -\sqrt[3]{26} \sim -2{,}9$ $f(0) = -26$

limiti: $-\infty, -1^{\pm}, 1^{\pm}, +\infty$ $\lim_{x \to \mp\infty} f(x) = \mp\infty$

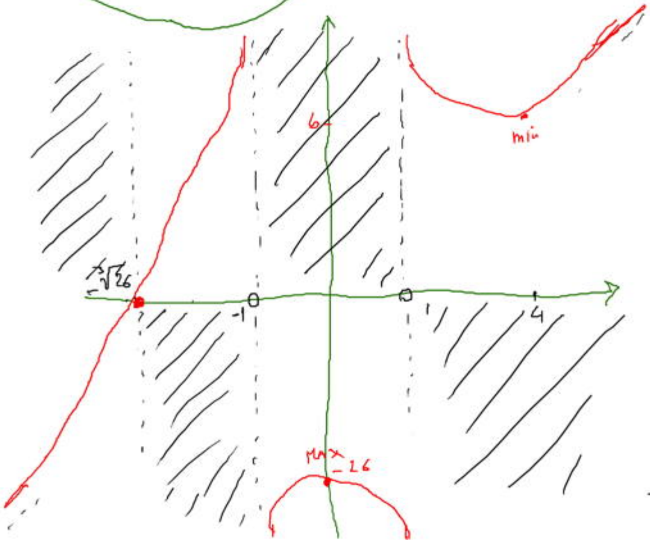
$\lim_{x \to -1^{\mp}} f(x) = \pm\infty$ $\lim_{x \to 1^{\pm}} f(x) = \pm\infty$

$f(x) > 0$ $N(x) = x^3+26 > 0$ $x^3 > -26$ $x > -\sqrt[3]{26}$

$D(x) = x^2-1 > 0$ $x^2 > 1$ $|x| > 1$ $x < -1 \vee x > 1$

		$-\sqrt[3]{26}$	-1	1
N	$-$	$+$	$+$	$+$
D	$+$	$+$	$-$	$+$
f	$-$	$+$	$-$	$+$

asintoti



$f'(x) = \dfrac{3x^2(x^2-1) - 2x(x^3+26)}{(x^2-1)^2} = \dfrac{x(x^3-3x-52)}{(x^2-1)^2}$

$f'(x) = 0$ $x = 0$ $x^3-3x-52 = 0$ $x = 4$

	1	0	-3	-52
4		4	16	52
	1	4	13	//

$(x-4)(x^2+4x+13) = 0$

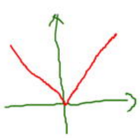
$f'(x) > 0$ $x < 0$, $x > 4$

		-1	0	1	4
f'	$+$	$+$	$-$	$-$	$+$
f	↗	↗	↘	↘	↗

$x = 0$ PTO MAX LOC $f(0) = -26$

$x = 4$ PTO MIN LOC $f(4) = 6$

$f(x) = |x|$



$x = 0$ è pto di min assoluto!

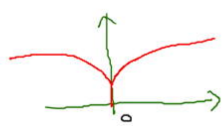
$f'(x) = \text{sign}(x) = \dfrac{|x|}{x} = \begin{cases} +1 & \text{se } x > 0 \\ -1 & \text{se } x < 0 \end{cases}$ $f'(x) = 0$ mai!

Per $f(x) = |x|$, il punto $x = 0$ è di min assoluto, ma è un pto SINGOLARE (in cui f NON È DERIVABILE)

Dunque va studiato "a parte"

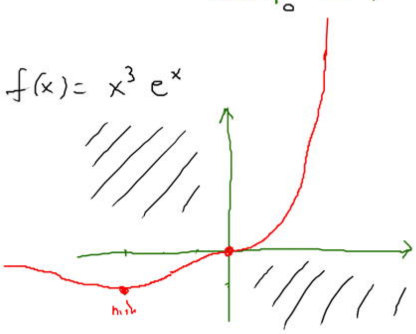
		0	
f'	$-$		$+$
f	↘	min ass.	↗

$f(x) = \sqrt{|x|}$



f non è derivabile in $x = 0$ $f'(x) = \dfrac{1}{2\sqrt{|x|}} \cdot \text{sign}\, x = \begin{cases} \dfrac{1}{2\sqrt{x}} & x > 0 \\ -\dfrac{1}{2\sqrt{|x|}} & x < 0 \end{cases}$

Ex. $f(x) = x^3 e^x$



Dominio $\mathbb{R}$ $f(x)$ non ha parità né periodo

$f(x) = 0$ $x = 0$ $f(x) > 0$ $x > 0$

$\lim_{x \to -\infty} f(x) = \lim_{t \to +\infty} (-t)^3 e^{-t} = \lim_{t \to +\infty} \left(-\dfrac{t^3}{e^t}\right) = 0^-$ ($t = -x$)

$\lim_{x \to +\infty} f(x) = +\infty$ $f'(x) = 3x^2 e^x + x^3 e^x = x^2(x+3)e^x$

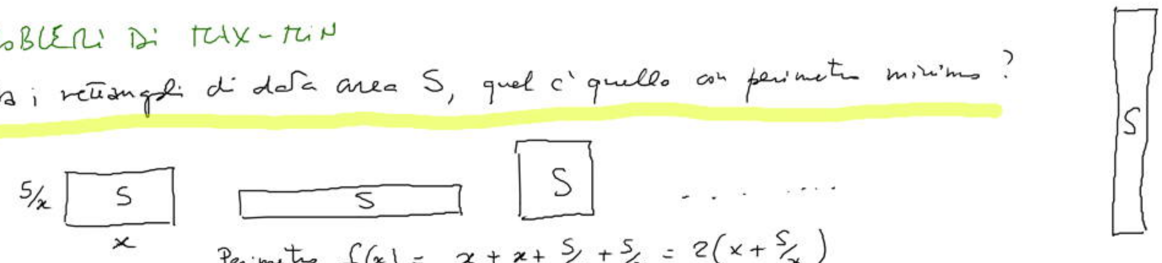
$f'(x) = 0$ $x = 0$, $x = -3$ $f'(x) > 0$ $x > -3$

$f(-3) = -\dfrac{27}{e^3} \sim -1{,}3$

		-3		0	
f'	$-$		$+$		$+$
f	↘		↗		↗

PROBLEMI DI MAX-MIN

- Tra i rettangoli di data area S, qual è quello con perimetro minimo?

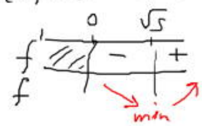


$\boxed{x>0}$

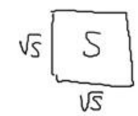
Perimetro $f(x) = x + x + S/x + S/x = 2(x + S/x)$

$f'(x) = 2\left(1 + S \cdot \left(-\frac{1}{x^2}\right)\right) = 2\left(1 - S/x^2\right)$ $\quad f'(x) = 0 \quad x = \sqrt{S}$

$f'(x) > 0 \quad \cancel{2}(1 - S/x^2) > 0 \quad x^2 - S > 0 \quad x > \sqrt{S}$



QUADRATO



- Un agricoltore deve scavare nel terreno una vasca a forma di piramide retta con base quadrata che contenga un dato volume V di acqua. Per impermeabilizzare la vasca deve acquistare del linoleum. Quale sarebbe la lunghezza del bordo della vasca che gli permetterebbe di risparmiare al massimo nell'acquisto di linoleum?

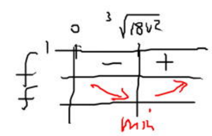
$f(x)$ = superficie da rivestire (va resa minima)

x = lato della piscina $\quad V = \frac{x^2 \cdot h}{3} \quad h = \frac{3V}{x^2}$

$$f(x) = 4\left(\frac{1}{2} \cdot x \sqrt{h^2 + \left(\frac{x}{2}\right)^2}\right) = 2x\sqrt{\frac{9V^2}{x^4} + \frac{x^2}{4}} = 2x\sqrt{\frac{36V^2 + x^6}{4x^4}} = \frac{\sqrt{x^6 + 36V^2}}{x}$$

$$f'(x) = \frac{\frac{1}{\cancel{2}}(x^6 + 36V^2)^{-1/2} \cdot (\cancel{6}^{\,3}x^5) \cdot x - 1 \cdot \sqrt{x^6 + 36V^2}}{x^2} = \frac{3x^6 - (x^6 + 36V^2)}{x^2\sqrt{x^6 + 36V^2}} = \frac{2(x^6 - 18V^2)}{x^2\sqrt{x^6 + 36V^2}}$$

$f'(x) = 0 \quad x^6 - 18V^2 = 0 \qquad \underline{x = \sqrt[3]{18V^2}} \qquad f'(x) > 0 \quad x > \sqrt[3]{18V^2}$



$f(\sqrt[3]{18V^2})$ è il linoleum da acquistare.

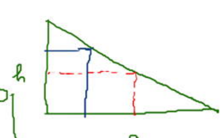
Ex (per la prossima volta)

- Studiare crescenza e estremi locali delle seg. funzioni:

  $f(x) = |x^2 - 2x| + |x - 1| \qquad f(x) = \log(x + e^{-x}) \qquad f(x) = \sqrt{3}\cos x - \sin x - 1$

- Risolvere i seg. problemi di max/min:

  (1) Dato un triang. rett. di base a e alt. h, qual è il rettangolo in esso inscritto di area massima?

  (2) Tra le pentole cilindriche di data sup esterna S, qual è la più capiente?