Box 13-3 THE DIRAC DELTA FUNCTION

We wish to demonstrate that the following integral is a representation of the one-dimensional Dirac delta function:

$$\delta(x - x') = \int_{-\infty}^{\infty} e^{2\pi i(x - x')S}\, dS$$

The results can easily be generalized to three dimensions. If this is the delta function, it must obey three properties.

First, if $x' = x$, then $\delta(x - x') = \infty$. It is obvious that, with $x = x'$, the exponential in the above integral is just unity; therefore, the integral is infinite.

Second, if $x' \neq x$, then $\delta(x - x') = 0$. It is not so obvious that the integral meets this requirement. The way to realize that it does is to note that the complex exponential is a periodic function that continually oscillates from -1 to 1 throughout all space. For each positive lobe there exists an adjacent (absolutely equivalent) negative lobe. The areas underneath these lobes cancel identically.

Third, if x' lies between a and b, then

$$\int_b^a dx\, \delta(x - x') = 1$$

Let $a = x' + \varepsilon$, and $b = x' - \varepsilon$. Then the area under the delta function is

$$\begin{aligned}\int_{x'-\varepsilon}^{x'+\varepsilon} dx \int_{-\infty}^{\infty} e^{2\pi i(x-x')S}\, dS &= \int_{-\infty}^{\infty} dS \int_{x'-\varepsilon}^{x'+\varepsilon} dx\, e^{2\pi i(x-x')S} \\ &= \int_{-\infty}^{\infty} e^{-2\pi i x'S}\, dS \int_{x'-\varepsilon}^{x'+\varepsilon} e^{2\pi i x S}\, dx \\ &= \int_{-\infty}^{\infty} e^{-2\pi i x'S}[(1/2\pi i S)(e^{2\pi i(x'+\varepsilon)S} - e^{2\pi i(x'-\varepsilon)S})]\, dS\end{aligned}$$

$$\begin{aligned}&= \int_{-\infty}^{\infty} e^{-2\pi i x'S}[(e^{2\pi i x'S}/2\pi i S)2i \sin 2\pi\varepsilon S]\, dS \\ &= (1/\pi) \int_{-\infty}^{\infty} [(\sin 2\pi\varepsilon S)/S]\, dS = 1\end{aligned}$$

because

$$\int_0^{\infty} [(\sin x)/x]\, dx = \int_{-\infty}^{0} [(\sin x)/x]\, dx = \pi/2$$

If x' is not between a and b, then the integral $\int_a^b dx\, \delta(x - x')$ is zero, because the function is everywhere zero. Thus we see that the integral originally given meets all the requirements, and is in fact the Dirac delta function.

A most important property of the delta function is the ability to shift the location of another function:

$$\int_{-\infty}^{\infty} dx\, f(x)\, \delta(x - x') = f(x')$$

We can demonstrate this by choosing a narrow interval $x' - \varepsilon$ to $x' + \varepsilon$ near x' and breaking up the integral into three parts:

$$\int_{-\infty}^{x'-\varepsilon} dx\, f(x)\, \delta(x - x') + \int_{x'-\varepsilon}^{x'+\varepsilon} dx\, f(x)\, \delta(x - x') + \int_{x'+\varepsilon}^{\infty} dx\, f(x)\, \delta(x - x')$$

The first and third integrals are zero for any finite-valued function $f(x)$, because everywhere within them $\delta(x - x') = 0$. The second integral can be evaluated if we choose ε small enough so that $f(x) = f(x')$; then it becomes

$$f(x') \int_{x'-\varepsilon}^{x'+\varepsilon} dx\, \delta(x - x') = f(x')$$