— Lectures on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

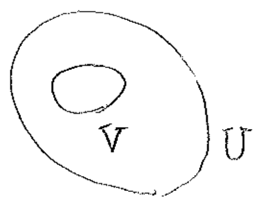
Mauro Spera

Lecture XIII

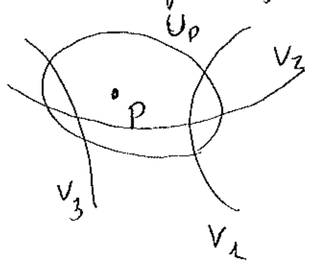
In Differential geometry one needs assembling global objects starting from local ones. This can be achieved by the so-called (smooth) partitions of unity. We need a topological detour.

Def. (paracompactness). (J. Dieudonné) A topological space X is said to be paracompact if any open cover $\mathcal{U}$ of X admits a locally finite refinement $\mathcal{V}$ (an open cover as well)

refinement: any $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$



locally finite: $\forall\, p \in X$, $\exists\, U_p \ni p$ (neighbourhood) intersecting only a finite number of V's in $\mathcal{V}$



Trivially, X compact $\Rightarrow$ X paracompact

(the converse is obviously false: think of $\mathbb{R}^n$, $n \geq 1$)

We state the following theorem, without proof

Th: Let X be locally compact (i.e. any point admits a neighbourhood with compact closure), Hausdorff, with countable basis. Then any $\mathcal{U}$ admits an at most countable locally finite refinement $\mathcal{V}$.

Therefore, X is then paracompact.

Smooth manifolds (and CW-complexes) turn out to
see Topogeo
be paracompact.

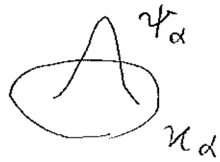
Let X be a topological space, $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in \Omega}$ an open cover of X.

A partition of unity subordinate to $\mathcal{U}$ is, by definition, a family of continuous functions

$$\{\psi_\alpha : M \longrightarrow \mathbb{R}\}_{\alpha \in \Omega}$$

such that:

1. $0 \leq \psi_\alpha \leq 1 \qquad (\forall \alpha \in \Omega, x \in X)$
2. $\underbrace{\text{supp}}_{\text{support}}\, \psi_\alpha \; (= \overline{\{x \in X / \psi_\alpha(x) \neq 0\}}) \subset \mathcal{U}_\alpha$ closure



3. $\{\text{supp}\, \psi_\alpha\}_{\alpha \in \Omega}$ is locally finite
4. $\sum\limits_{\alpha \in \Omega} \psi_\alpha(x) = 1 \qquad \forall x \in X$ ← This property justifies the name

   ↖ a finite sum is involved at each point

One finds that a Hausdorff topological space X is
$$\boxed{\text{paracompact} \iff \text{every open cover of } X \text{ admits a partition of unity subordinate to it.}}$$

Notice that $(\Leftarrow)$ is trivial: take $V_\alpha \doteq \{x \in X / \psi_\alpha(x) \neq 0\}$, their collection yields a locally finite refinement of $\{\mathcal{U}_\alpha\}$.

We treat the following simple but instructive case:

Let M be a compact (smooth) manifold, equipped with a finite atlas (this can be achieved in view of compactness). We are going to construct a <u>smooth</u> partition of unity subordinate to it.

Let $\mathcal{A} = \{\mathcal{U}_i, \varphi_i\}_{i=1..N}$ be the atlas in question — some N

First of all, we may alter the local charts φ_i in such a way that

$$\varphi_i : \mathcal{U}_i \overset{\text{homeo}}{\xrightarrow{\ \sim\ }} B_1(0) \subset \mathbb{R}^n$$

ball of radius 1 centred at 0

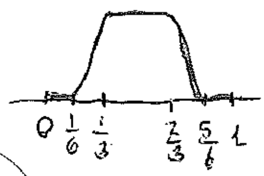
Define:

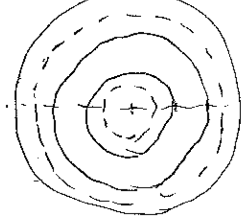
$$W_i \subset V_i \subset \mathcal{U}_i \qquad i = 1..N$$

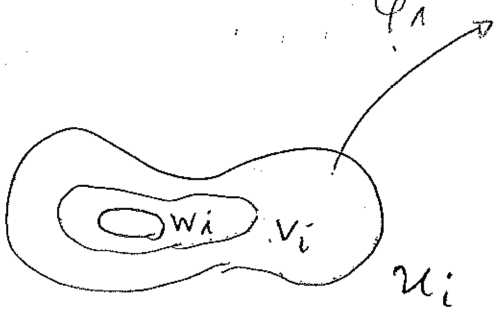
$$W_i := \varphi_i^{-1}\left(B_{\frac{1}{3}}(0)\right)$$

radius

$$V_i := \varphi_i^{-1}\left(B_{\frac{2}{3}}(0)\right)$$







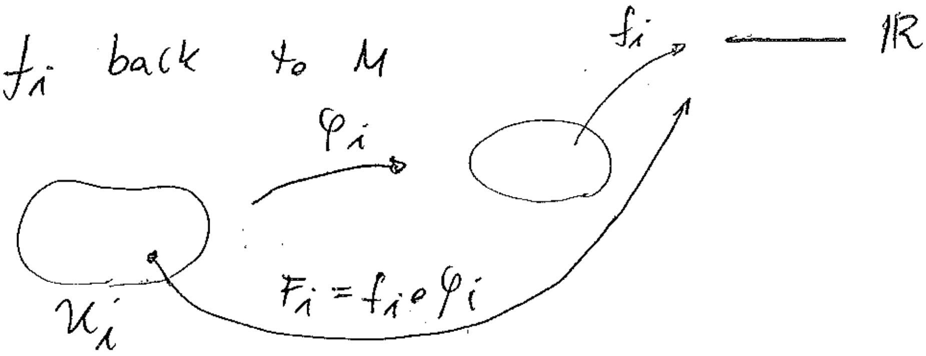
Now let $f_i : \mathbb{R}^n \to \mathbb{R}$ a bump function adapted to

they can be equal to a single function f

see below, auxiliary constructions

Set $F_i = \varphi_i^* \circ f_i \quad (= f_i \circ \varphi_i)$

That is, pull f_i back to M



F_i is smooth $\quad (F_i \circ \varphi_i^{-1} = f_i \circ \varphi_i \circ \varphi_i^{-1} = f_i$ is smooth $) \ldots$

Now set $\quad \psi_i := \dfrac{F_i}{\sum_{i=1}^{N} F_i}$

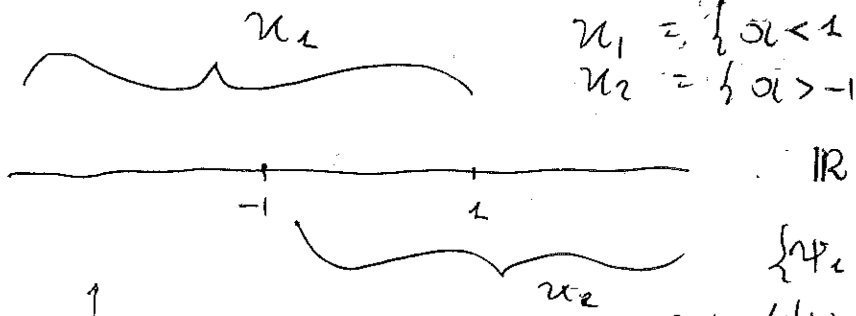
$\neq 0$ since every $x \in M$ belongs to some $\mathcal{U}_i$

★ The $\{\psi_i\}_{i=1,2..N}$ is the sought for smooth partition of unity subordinate to $\{\mathcal{U}_i\}_{i=1..N}$

Indeed the properties $0 \leq \psi_i \leq 1$, $\operatorname{supp} \psi_i \subset \mathcal{U}_i$,

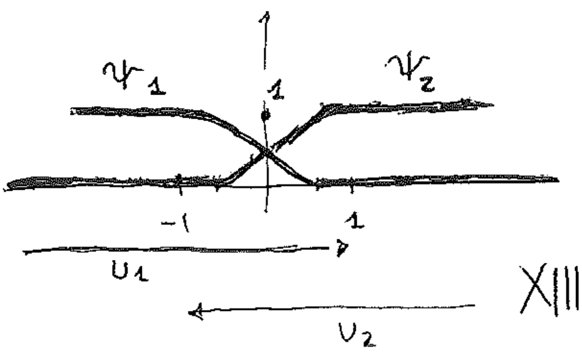
and $\displaystyle \sum_{i=1}^{N} \psi_i = \sum_{i=1}^{N} \frac{F_i}{\sum_{j=1}^{N} F_j} = 1$ are obvious.

Example



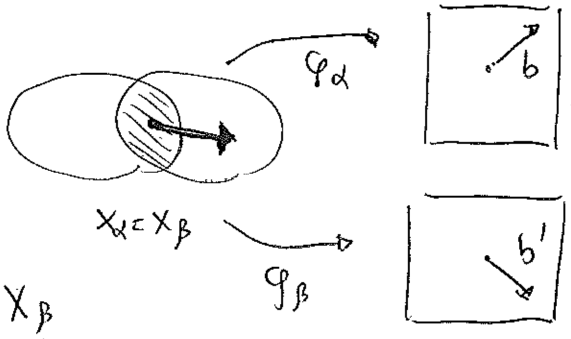
$\mathcal{U}_1 = \{x < 1\}$
$\mathcal{U}_2 = \{x > -1\}$

$\{\psi_1, \psi_2\}$ is a partition of unity subordinate to $\{U_1, U_2\}$



Let us now turn to the construction of global objects, given local ones.

★ vector fields. Let X_α, $\alpha \in \mathfrak{A}$, be local vector fields on $\mathcal{U}_\alpha$ fulfilling $X_\alpha = X_\beta$ if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$



b & b' related by the correct transformation law

X_α & X_β coincide on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$, namely

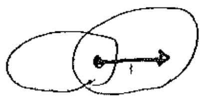
Then, if $\{\rho_\alpha\}_{\alpha \in \mathfrak{A}}$ is a smooth partition of unity subordinate to $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in \mathfrak{A}}$, set

$$\boxed{X = \sum_\alpha \rho_\alpha X_\alpha}$$

notice that at each point x, the sum reduces to a finite one! Consistency is assured in view of $\sum \rho_\alpha = 1$

$$X(x) = X_\alpha(x) = X_\beta(x)$$

if $x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$

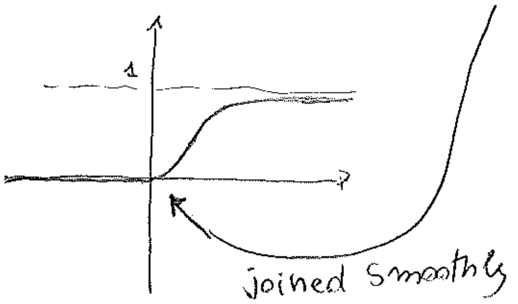


$X = X_\alpha = X_\beta$

★ One can similarly "glue" local forms, metrics, generic tensors. Obviously, a global object induces local objects subject to the appropriate transformation laws.

★ Auxiliary constructions

Let: $f : \mathbb{R} \to \mathbb{R}$ $\quad f(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0 \end{cases}$



★ f is smooth but not analytic

$f(t) \neq 0 = \sum_{k=0}^{\infty} \frac{\overbrace{f^{(k)}(0)}^{0}}{k!} t$

f is smooth (enough to check it at 0)

First of all, f is continuous $\left(\lim_{t \to 0^+} \frac{e^{-\frac{1}{t}}}{t^k} = 0 \right.$ for $\left. k > 0 \right)$

If $t > 0$, then $\underbrace{f^{(k)}}_{k\text{th-derivative}}(t) = \frac{\overbrace{P_k(t)}^{\text{polynomial}}}{t^{2k}} e^{-\frac{1}{t}}$

(perform induction...) and $\lim_{t \to 0^+} f^{(k)}(t) = 0$

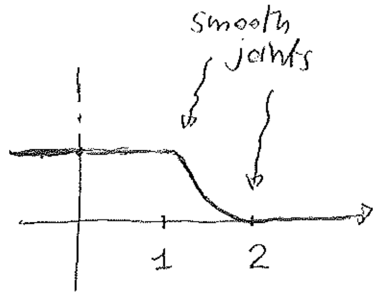
$\Rightarrow$ $f^{(k)}(0) = 0$ $\qquad$ $f^{(k)}$ is continuous $\forall k$,

they exist $\qquad$ so it is smooth.

Now take

$h(t) = \frac{f(2-t)}{f(2-t) + f(t-1)}$

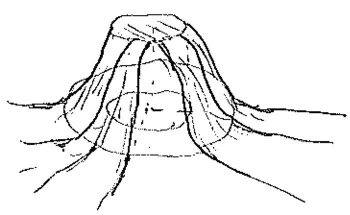
cutoff function



In $\mathbb{R}^n$ $(n \geq 1)$, $\mathbf{H} = h(\|x\|)$ satisfies $\quad 0 \leq H(x) \leq 1$

★ bump function $\qquad H \equiv 1$ on $\overline{B}_1(0)$

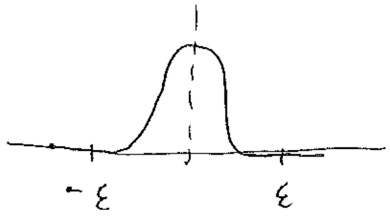
$\operatorname{supp} H = \overline{B}_2(0)$



Another example also called mollifier

$$f_\varepsilon(x) = \begin{cases} C_\varepsilon\, e^{-\frac{\varepsilon^2}{\varepsilon^2 - x^2}} & |x| < \varepsilon \\ 0 & |x| \geq \varepsilon \end{cases}$$

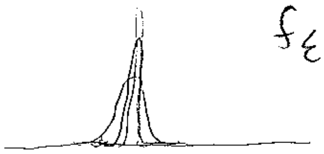
C_ε is chosen in such a way that $\int_{\mathbb{R}} f_\varepsilon(x)\,dx = 1$



In $\mathbb{R}^n$ $n \geq 1$

one finds, similarly:

$$f_x(x) = \begin{cases} C_\varepsilon\, e^{-\frac{\varepsilon^2}{\varepsilon^2 - \|x\|^2}} & \|x\| < \varepsilon \\ 0 & \|x\| \geq \varepsilon \end{cases}$$



$f_\varepsilon \xrightarrow[\mathcal{D}']{} \delta$

as distributions

(cf course in Functional Analysis)