Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture II

★ Dual (or adjoint) homomorphism

Let $T \in \mathrm{Hom}(V, W)$ $\quad \dim_K V = n \quad \dim_K W = m$

T: homomorphism, linear map, linear transformation, linear operator

operations in V, K

$$\left[\; T(\alpha \cdot v_1 + \beta \cdot v_2) = \alpha \cdot T v_1 + \beta \cdot T v_2 \;\right]$$

operations in W

Let $e = (e_1 \dots e_n)$, $f = (f_1 \dots f_m)$ bases in V and W, respectively

The $m \times n$ matrix $\quad m_{fe}(T) = (t_{ik})_{\substack{i=1..m \\ k=1..n}} \quad$ representing T

(f: final, e: initial)

with respect to the given bases is

$$m \text{ rows} \left\{ \underbrace{\begin{pmatrix} & & \\ & t_{ik} & \\ & & \end{pmatrix}}_{\substack{Te_k \\ n \text{ columns}}} \right. \leftarrow i$$

that is

$$Te_k = \sum_{i=1}^{m} t_{ik} f_i$$

and, clearly

$$\boxed{t_{ik} = f_i^*(Te_k)} \qquad (\diamond)$$

f_i^*: ith -vector of the dual basis $f^* = (f_1^* \dots f_m^*)$ of W^*

The adjoint (or dual) homomorphism T' (of T) is an element in $\mathrm{Hom}(W^*, V^*)$

caveat!

$T: V \to W$

$T': W^* \to V^*$ defined as

$$\boxed{\underbrace{(T' \underset{W^*}{\underset{\Uparrow}{l}})}_{\Uparrow \atop V^*}(\underset{V}{\underset{\Uparrow}{v}}) := \underset{W^*}{\underset{\Uparrow}{l}}(\underset{W}{\underset{\Uparrow}{Tv}})} \quad \forall v \in V \quad \forall l \in W^*$$

Any matrix representation of T' will be of type $n \times m$; in particular, if $e^* = (e_1^*, \dots, e_m^*)$, $f^* = (f_1^* \dots f_m^*)$ denote the dual bases of e, f, respectively, one finds

$$\boxed{m_{\underset{\text{final}}{e^*}\underset{\text{initial}}{f^*}}(T') = m_{fe}(T)^t}$$

(This provides an intrinsic meaning to the notion of transpose of a matrix)

Pf. $$[m_{e^*f^*}(T')]_{ik} \overset{\text{recall}(\diamond)}{=} e_i^{**}(T' f_k^*) = (T' f_k^*)(e_i)$$

recall $x^{**}(y^*) = y^*(x)$

$$\underset{\text{by definition of } T'}{=} f_k^*(Te_i) \underset{(\diamond)\text{ again}}{=} t_{ki}$$

This concludes the proof $\square$

Example $\quad$ Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ $\leftarrow$ columns

$$T \equiv m_{ee}(T) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 5 \end{pmatrix}$$

canonical basis

we show that $T' = T^t$ using the definition

$T' : \mathbb{R}^3 \to \mathbb{R}^2$ $\leftarrow$ rows

$\ell : \mathbb{R}^3 \to \mathbb{R} \quad \ell \leftrightarrow (x, y, z)$

$$(T'\ell)\Big(\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{e_1}\Big) := \ell(Te_1) = (x, y, z)\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = x + 2y$$

$$(T'\ell)\Big(\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{e_2}\Big) := \ell(Te_2) = (x, y, z)\begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix} = x + 4y + 5z$$

Therefore

$$(x \;\; y \;\; z) \xmapsto{T'} (x + 2y, \; x + 4y + 5z)$$

$$\| $$

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 1 & 4 & 5 \end{pmatrix}}_{T^t} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

* Transformation laws (contravariance & covariance)

(I) Let $T \in GL(V)$ $\qquad T: V \to V$

invertible endomorphisms

notation: general linear group associated to V

The problem is the following: find $S': V^* \to V^*$, $S' \in GL(V^*)$

(stemming from $S: V \to V$, $S \in GL(V)$)

such that, $\forall \ell \in V^*$, $\forall v \in V$

$$\boxed{(S'\ell)(Tv) = \ell(v)}$$

that is, if vectors in V are transformed via T, how should vectors in V^* transform, in order that the corresponding evaluations do not change?

["Gattopardo" Principle]

One immediately finds, given $\ell \in V^*$

$$(S'\ell)(Tv) = \ell(STv) = \ell(v) \quad \forall v \in V$$

if and only if $\quad ST = I \Rightarrow S = T^{-1}$

$\Rightarrow S' = (T^{-1})'$ (which is easily seen to equal $(T')^{-1}$)

The upshot is that

to $\quad T: V \to V \qquad$ "contravariance"

there corresponds $\quad (T')^{-1} = (T^{-1})' \qquad$ "covariance"

vectors in V: contravariant vectors $\qquad$ "vectors"

V^*: covariant vectors $\qquad$ "covectors"

vectors from V and V^* can be distinguished
"vectors" "covectors"
by their behaviour under linear transformations.

II Let us deal with the same problem from a slightly different, more concrete standpoint

Again consider the scalar $\underset{V^*}{l}(\underset{V}{v})$

fix bases $e=(e_1 \dots e_n)$ and $e'=(e_1' \dots e_n')$.

together with the corresponding dual bases.

Then $v = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i' e_i'$

and one has

$$x' = \overbrace{M_{e'}^{e}(I)}^{\text{change of basis matrix}} x$$

$$M_{e'}^{e}(I) \equiv A \in GL(n, K) \quad \text{nonsingular}$$

Similarly, one has

$$l = \sum_{i=1}^{n} y_i e_i^* = \sum_{i=1}^{n} y_i' e_i'^*$$

and, obviously

$$y' = B y$$

for some $B \in GL(n, K)$

what is then B?

We have, successively

$$l(v) = y^t x = y'^t x'$$

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$v \leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x$

$l \leftrightarrow y^t = (y_1 \dots y_n)$

$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

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But $y^t x = y^t A^{-1} A x$

$= (A^{-t} y)^t x'$

$\Rightarrow (A^{-t} y)^t x' = y'^t x'$

$\forall x' \in \mathbb{R}^n$

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$$\Rightarrow y' = A^{-t} y \quad \text{i.e. } B = A^{-t}$$

Therefore $v \mapsto \overset{\overset{A}{\parallel}}{B^{-t}} v$

$l \mapsto \underset{\underset{A^{-t}}{\parallel}}{B}\, l$

This is of course consistent with the previous discussion.

III Still another point of view "vintage" definition

Let $e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}$ ← vectors

$v = \sum x_i e_i \equiv x^t e \qquad x^t = (x_1 \dots x_n)$

$= \sum x_i' e_i' \equiv x'^t e' \qquad \boxed{\quad x^t \quad} \boxed{e}$

Then, if $\underset{\in\, GL(n,\mathbb{R})}{x' = Ax}$ $\qquad \underset{x'}{[\;]} = \underset{A}{[\quad]}\,\underset{x}{[\;]}$

Covariance, i.e coordinate like transformation
coordinates transform covariantly

Then $e' = A^{-t} e$

i.e bases transform contravariantly

Indeed:

$x'^t e' = (Ax)^t e' = x^t A^t e' \qquad \forall x \in \mathbb{R}^n$

$\quad \|$

$x^t e$

$\Rightarrow \quad e = A^t e', \quad e' = A^{-t} e$

Covariance: coordinate type transformation

Contravariance: basis type transformation

* Important example

Let $\underline{r} = \underline{r}(u,v)$ $\quad (u,v) \in \underbrace{\mathcal{U}}_{\text{region}} \subset \mathbb{R}^2$ $\quad \underline{r} \in \mathcal{C}^6$

parametric surface

(This concept is invariant under regular parameter changes)

$$\begin{cases} u' = u'(u,v) \\ v' = v'(u,v) \end{cases} \qquad (u,v) \in \mathcal{U}$$

the differential of $(u,v) \overset{f}{\longmapsto} (u',v')$ reads at a generic point:

$$\begin{cases} du' = \dfrac{\partial u'}{\partial u} du + \dfrac{\partial u'}{\partial v} dv \\[2ex] dv' = \dfrac{\partial v'}{\partial u} du + \dfrac{\partial v'}{\partial v} dv \end{cases} \quad \text{i.e.} \quad \begin{pmatrix} du' \\ dv' \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial u'}{\partial u} & \dfrac{\partial u'}{\partial v} \\[2ex] \dfrac{\partial v'}{\partial u} & \dfrac{\partial v'}{\partial v} \end{pmatrix}}_{J:\ \text{Jacobian matrix}} \begin{pmatrix} du \\ dv \end{pmatrix}$$

One has a corresponding transformation (basis change) on the tangent plane (at any point)

$$\begin{cases} \dfrac{\partial \underline{r}}{\partial u'} = \dfrac{\partial \underline{r}}{\partial u}\dfrac{\partial u}{\partial u'} + \dfrac{\partial \underline{r}}{\partial v}\dfrac{\partial v}{\partial u'} \\[2ex] \dfrac{\partial \underline{r}}{\partial v'} = \dfrac{\partial \underline{r}}{\partial u}\dfrac{\partial u}{\partial v'} + \dfrac{\partial \underline{r}}{\partial v}\dfrac{\partial v}{\partial v'} \end{cases} \qquad \begin{pmatrix} \dfrac{\partial \underline{r}}{\partial u'} \\[2ex] \dfrac{\partial \underline{r}}{\partial v'} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial u'} & \dfrac{\partial v}{\partial u'} \\[2ex] \dfrac{\partial u}{\partial v'} & \dfrac{\partial v}{\partial v'} \end{pmatrix} \begin{pmatrix} \dfrac{\partial \underline{r}}{\partial u} \\[2ex] \dfrac{\partial \underline{r}}{\partial v} \end{pmatrix}$$

abstractly:

$$\begin{pmatrix} \dfrac{\partial}{\partial u'} \\[2ex] \dfrac{\partial}{\partial v'} \end{pmatrix} = \underbrace{\begin{pmatrix} \dfrac{\partial u}{\partial u'} & \dfrac{\partial v}{\partial u'} \\[2ex] \dfrac{\partial u}{\partial v'} & \dfrac{\partial v}{\partial v'} \end{pmatrix}}_{J^{-t}} \begin{pmatrix} \dfrac{\partial}{\partial u} \\[2ex] \dfrac{\partial}{\partial v} \end{pmatrix}$$

directional derivatives along coordinates

remember $J_{f^{-1}} = J_f^{-1}$

From $\dfrac{\partial x}{\partial x} = \dfrac{\partial v}{\partial v} = 1 \quad , \quad \dfrac{\partial x}{\partial v} = \dfrac{\partial v}{\partial x} = 0$

we conclude that, setting

$$e = \left(\frac{\partial}{\partial x} , \frac{\partial}{\partial v} \right)$$

then $\quad e^* = (dx , dv)$

We shall resume this discussion later on.