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Master's Program in Applied Mathematics

Written test of Functional Analysis

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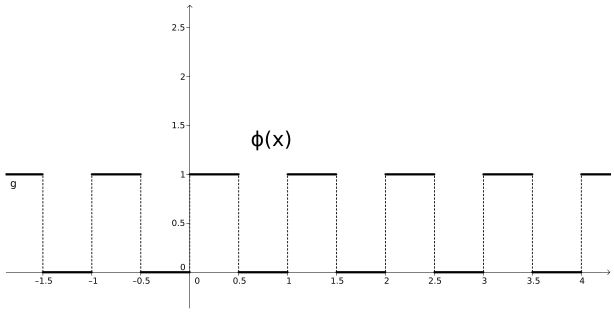
Solve some of the following problems. Justify your conclusions. Time: 120 min.

Pb 1. Consider the 1-periodic function $\varphi : \mathbb{R} \to \mathbb{R}$ whose restriction to $[0, 1]$ is $\mathbf{1}_{[0,1/2]}(x)$.



Study the strong and weak convergence in $L^2([0, 1])$ of the sequence $\varphi_n(x) = \varphi(nx)$.

Pb 2. Let $Y = \{(x_n) \in \ell^6 : x_1 = x_3 = x_5 = \cdots = 0\}$ and let $\varphi \in Y'$. Prove that there is a unique linear extensions $\Phi : \ell^6 \to \mathbb{R}$ of φ with $\|\Phi\|_{(\ell^6)'} = \|\varphi\|_{Y'}$.

Pb 3. Let $\{K_n\}_{n\in\mathbb{N}}$ be a sequence of nonempty, closed and convex subsets of an Hilbert space H such that $K_n \supseteq K_{n+1}$ for all $n \in \mathbf{N}$. Assume that $K_\infty := \bigcap_{n\in\mathbb{N}} K_n$ is nonempty: prove that it is a closed, convex set. Denote by $\pi_j : H \to K_j$ the projection map for $j \in \mathbf{N} \cup \{\infty\}$. Prove that for every $x \in H$ we have $\lim_{n\to\infty} \pi_n(x) = \pi_\infty(x)$. (*HINT: The sequence $\|x - \pi_n(x)\|$ is non-decreasing and bounded. Apply the parallelogram law to the vectors $x - \pi_n(x)$ and $x - \pi_m(x)$ $(m > n)$ to deduce that $\{\pi_j(x)\}_{j\in\mathbf{N}}$ is a Cauchy sequence in $H\ldots$*)

Pb 4. Prove that the operator $T : C([0, 1]) \to \ell^1$ defined by

$$(T(f))_n := a_n \int_0^{\frac{1}{n}} f(x)dx, \quad f \in C([0, 1]),$$

is compact whenever $\{\frac{a_n}{n}\} \in \ell^1$.

Pb 5. Prove that the set $C = \{f \in C^1([0, 1]) : f(0) = 0,\ \|f'\|_{L^8(0,1)} \le 1\}$ is contained in the Hölder space $C^{0;\frac{7}{8}}(0, 1)$. Then justify why C is relatively compact in $C([0, 1])$.

Pb 6. Consider the linear functional $T : W^{1,2}([0, 1]) \to \mathbb{R}$ such that $T(u) = u(0)$ whenever $u \in W^{1,2}([0, 1])$ (u the absolutely continuous representative). Prove that $T \in (W^{1,2}([0, 1]))'$. Consider then the map $S : W^{1,2}([0, 1]) \to L^2([0, 1])$ defined by $(Su)(x) := u(0) + u'(x)$. Is S linear, continuous, compact?