Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

Mauro Spera

Lecture XX

∗ On the Frobenius theorem

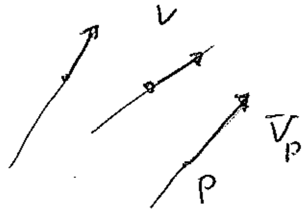
Prelude: Working in $\mathbb{R}^2$, for simplicity, the problem

$$\begin{cases} \dot{x} = X \\ \dot{y} = Y \end{cases} \qquad V = (X, Y) \quad \text{vector field}$$

consists in the following:

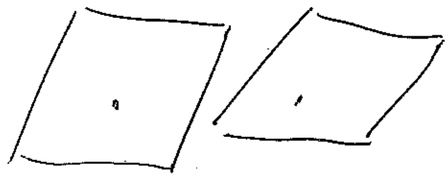
Given a direction field, determined, pointwise, by a vector field $\overline{V}$ (one has $\langle \overline{V}_p \rangle \leq \mathbb{R}^2$)

↑ subspace

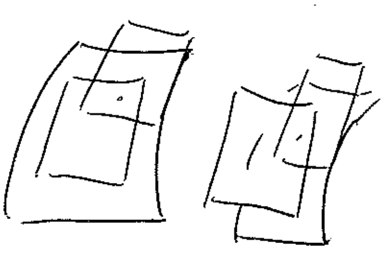


find its integral curves, i.e. curves such that whose tangents at each point have the given direction. The Cauchy - Lipschitz theorem asserts that, at least locally, the problem admits a (unique) solution.

Let us generalize. Work for instance in $\mathbb{R}^3$, and assign a distribution of planes $p \longmapsto \Delta_p$ in $\mathbb{R}^3$



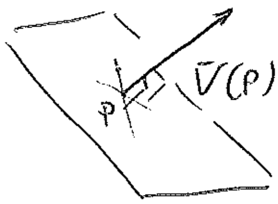
Does there exist, $\forall p$, an integral manifold of the above distribution, namely, a surface Σ_p such that $T_p \Sigma_p = \Delta_p \ \forall p$?



[in order to identify Δ_p, let us assign a pair of vector fields (X_1, X_2) such that, locally $(X_1(q), X_2(q))$ yield a basis of Δ_q:

$\mathrm{span}(X_1(q), X_2(q)) = \Delta_q$

To be concrete, let $V \in \mathfrak{X}(\mathbb{R}^3)$ be a vector field. Consider, at each point $p \in \mathbb{R}^3$, the plane π_p through p perpendicular to $\vec{V}(p)$.



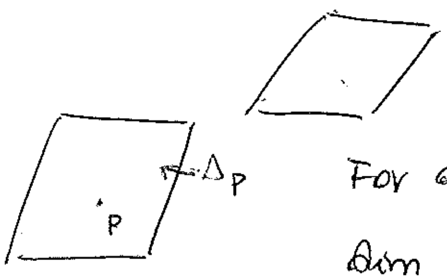
One obtains a distribution of planes. We are going to discuss a necessary and sufficient condition ensuring that $\pi_p = T_p \Sigma_p$

surface

In fluid mechanics, this is the issue of <u>laminarity</u>.

The answer is NO, in general. (†) See page XX-1'

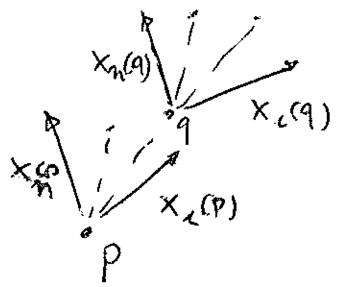
Let us formalize the preceding discussion



Let M be a smooth manifold, $\dim M = n + k$. For all $p \in M$, let $\Delta_p \leq T_pM$, $\dim \Delta_p = n$. Consider the map

$$\Delta : \quad M \ni p \longmapsto \Delta_p \leq T_pM$$

Assume that $\forall p \in M$, $\exists$ $U \ni p$ and $X_1, \ldots, X_n \in \mathcal{X}(U)$ such that, $\forall q \in U$, $(X_1(q) \ldots X_n(q))$ provide a basis for Δ_q. Δ is then termed smooth distribution of dimension n (smooth n-dimensional distribution) and $(X_1 \ldots X_n)$ gives a local basis thereof.



Def. Δ is called involutive (involutoria) if, in a suitable neighbourhood of each point, there exists a local basis $(X_1 \ldots X_n)$ of Δ such that, for suitable $C^k_{ij} \in \mathcal{C}^\infty(M)$, one has

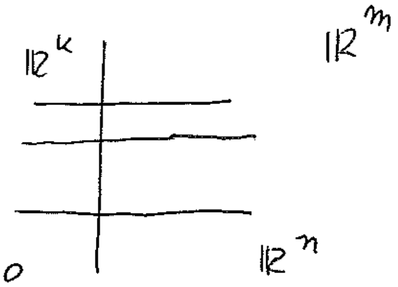
$$[X_i, X_j] = C^k_{ij} X_k \quad \leftarrow \text{Einstein}$$

Lie bracket

i.e. Δ is closed (locally) with respect to the Lie bracket of vector fields.

Typical example : in $\mathbb{R}^m$ $\qquad m = n + k$

$$\Delta = \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\rangle$$

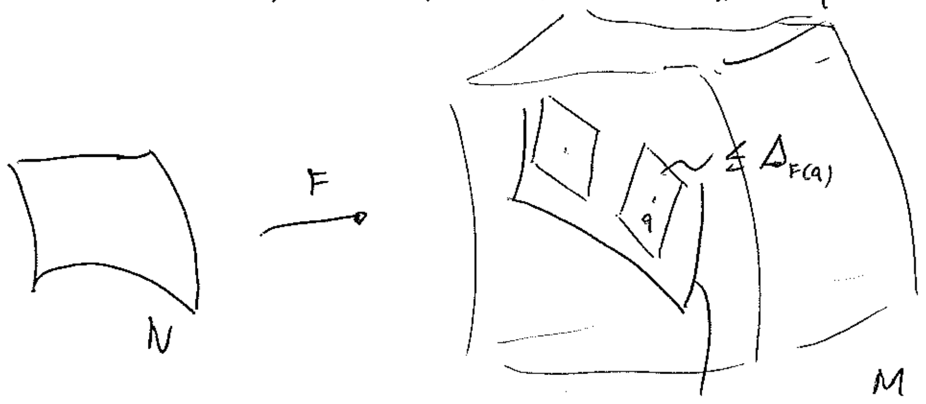


Notice that in this case

$$c^k_{ij} \equiv 0 \qquad \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0$$

The $\frac{\partial}{\partial x^i}$ commute

Def. (i) An integral submanifold of Δ is a manifold N such that, if $F : N \to M$ is an injective immersion, one has, $\forall q \in N$, $F_*(T_q(N)) \leq \Delta_{F(q)}$

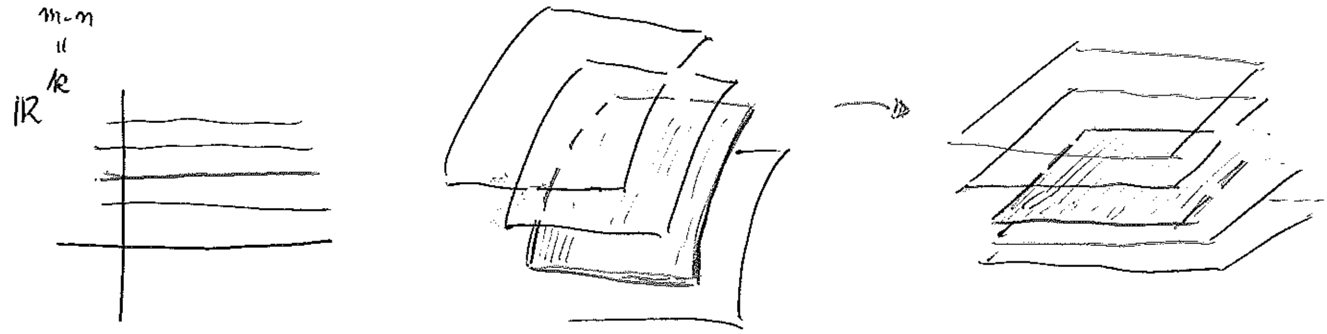


F(N)
(identified with N...)

(ii) Δ is called completely integrable if, $\forall p \, M$, $\exists$ a local coordinate system (defined on $U \ni p$) $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right)$ giving a local basis for Δ.

This being the case, there exists N integral submanifold with $T_q N = \Delta_q \quad \forall q \in \bar{U}$, defined by $\{ x^i = a^i, \; i = n+1, \dots m \}$

i.e. N is an <u>n-slice</u> of $\bar{U}$



Notice that Δ is clearly <u>involutive</u>.
The basic result governing the theory is the following:

** Theorem (Frobenius) Δ is <u>involutive</u> if and only if it is <u>completely integrable</u>

Comment: ($\Leftarrow$) is clear in view of the preceding discussion. Also notice that if $N \subset M$ is a submanifold of M, $\mathfrak{X}(N)$ is a Lie algebra. So the condition is necessary. The crucial point is to prove sufficiency, which can be ascertained by <u>induction</u> and using the fact that an involutive distribution can be generated pointwise by <u>commuting</u> vector fields.

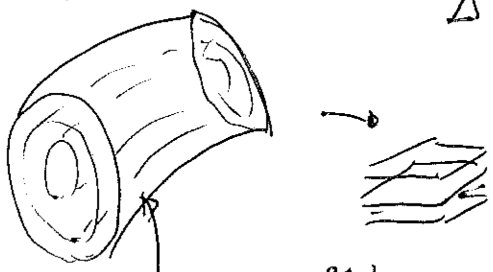
*** <u>Application</u>. The <u>Liouville - Arnol'd</u> theorem in mechanics.
Given a Hamiltonian system (M, ω, H) $\dim M = 2n$
$\{f_i\}_{i=1..n}$ first integrals in involution ($\{f_i, f_j\} = 0$ ($\Rightarrow [X_{f_i}, X_{f_j}] = 0$) which are also functionally independent, the X_{f_i} yield a completely integrable distribution, whose integral submanifolds, <u>under suitable conditions</u>, are tori (Liouville or Lagrangian tori)

one can find local action-angle variables

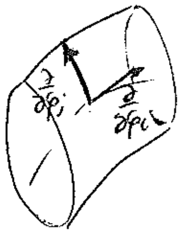
$N \sim (\varphi_1 \dots \varphi_n, I_1 \dots I_n)$
angles actions

$\omega = d\varphi \wedge dI \quad (= \sum d\varphi_i \wedge dI_i)$

$\Delta = \left\{ \frac{\partial}{\partial \varphi_i} \right\}_{i=1..n}$



torus labelled by I, with angle coordinates on it (Lagrangian, or Liouville torus)

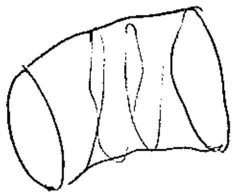


Hamilton's equations become:
and are indeed immediately integrated ...

$$\begin{cases} \dot{I}_i = 0 \Rightarrow I_i = \text{constant} \\ \dot{\varphi}_i = c_i \quad (\text{constant}) \end{cases}$$

$\varphi_i = \varphi_i^0 + t c_i$
linearization of the flow

<u>Example</u>: the harmonic oscillator.



the trajectory winds on a fixed Lagrangian (or Liouville) torus

Another very important example

Let G be a Lie group, with Lie algebra $\mathfrak{g}$

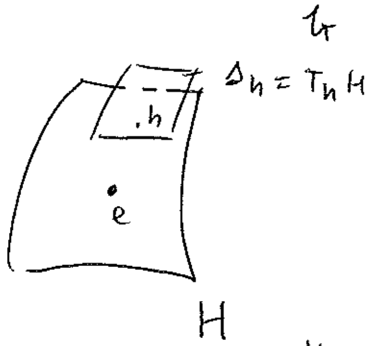
H a Lie subgroup of G, with Lie algebra $\mathfrak{h}$
(it is enough to require H closed, then it will be automatically a Lie group)

$$\mathfrak{h} = \left\{ \begin{array}{l} \text{left invariant vector fields on } G \\ \text{and tangent to } H \text{ in } \underset{\substack{\uparrow \\ \text{neutral element}}}{e} \end{array} \right\}$$

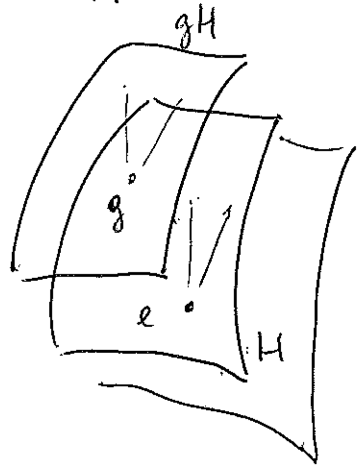
$\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$

So there exists Δ, distribution on G such that $\Delta_h = T_h H$ and, in general,

$$\Delta_g = T_g(gH)$$



By Frobenius' theorem, Δ is completely integrable and its integral submanifolds are precisely the cosets $gH = \{ gh \,/\, h \in H,\ g \in G \text{ fixed} \}$
Classi laterali



Note: if H is closed, gH is an embedded submanifold of G and the quotient space G/H will be a manifold as well.

* Integrability and differential forms

Let us apply, in a very special case, Cartan's calculus to the integrability problem, in order to illustrate its tremendous power.

Let us work in $\mathbb{R}^3$, and let a plane distribution be given; it can be described as the kernel of a 1-form θ: $\theta = 0$

$$\theta_1 dx + \theta_2 dy + \theta_3 dz = 0$$

In view of the Frobenius theorem, the above distribution is integrable $\Leftrightarrow$ there exist, in a neighbourhood of any point, local coordinates (ξ, η, ζ) such that the integral manifolds through every point of the neighbourhood have the form $\zeta = c$

The condition $\theta = 0$ translates to $f(\xi, \eta, \zeta)\, d\zeta = 0$ ($f \in C^\infty$). Now compute:

$$d\theta = \frac{\partial f}{\partial \xi} d\xi \wedge d\zeta + \frac{\partial f}{\partial \eta} d\eta \wedge d\zeta$$, and observe that

$$\theta \wedge d\theta = f\, d\zeta \wedge \left(\frac{\partial f}{\partial \xi} d\xi \wedge d\zeta + \frac{\partial f}{\partial \eta} d\eta \wedge d\zeta \right) = 0$$

$$\boxed{\theta \wedge d\theta = 0}$$

This condition is intrinsic, and translates this particular instance of Frobenius theorem. Indeed, it is also a sufficient condition for integrability

Reverting to Cartesian coordinates, one has:

$$(\theta_1 \, dx + \theta_2 \, dy + \theta_3 \, dz) \wedge \Big[\Big(\frac{\partial \theta_3}{\partial y} - \frac{\partial \theta_2}{\partial z} \Big) dy \wedge dz + \Big(\frac{\partial \theta_1}{\partial z} - \frac{\partial \theta_3}{\partial x} \Big) dz \wedge dx + \Big(\frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y} \Big) dx \wedge dy \Big] = 0$$

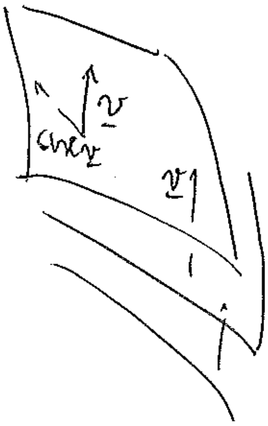
namely

$$\theta_1 \Big(\frac{\partial \theta_3}{\partial y} - \frac{\partial \theta_2}{\partial z} \Big) + \theta_2 \Big(\frac{\partial \theta_1}{\partial z} - \frac{\partial \theta_3}{\partial x} \Big) + \theta_3 \Big(\frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y} \Big) = 0$$

in vector analysis notation: $\theta \rightsquigarrow \underline{v}$

one gets $\boxed{\underline{v} \cdot \operatorname{curl} \underline{v} = 0}$

(velocity $\perp$ vorticity) : this ensures the <u>laminar</u> character of a fluid (which proceeds with "fronts")



Example $\quad w = x\,dy + dz$

kernel : $\quad (x\,dy + dz)\left(\alpha \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + \gamma \frac{\partial}{\partial z}\right) = 0$ $\quad$ (α, β, γ functions)

$\Rightarrow \quad x\beta + \gamma = 0 \quad \Rightarrow \quad \gamma = -\beta x$

$\Rightarrow$ a local basis for the distribution Δ is, for instance $\left(\frac{\partial}{\partial x}, \; \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right)$

Let us compute $\left[\frac{\partial}{\partial x}, \; \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right]$.

$$= \underbrace{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]}_{= 0} + \left[\frac{\partial}{\partial x}, -x\frac{\partial}{\partial z}\right]$$

$\ldots = \frac{\partial(-x)}{\partial x}\frac{\partial}{\partial z} = -\frac{\partial}{\partial z}$ which is not a linear combination (with function coefficients) of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$

shorthand computation! ⚠

$\Rightarrow$ Δ is not integrable. Let us obtain the same conclusion via Cartan's calculus (it is immediate)

$dw = dx \wedge dy \quad ; \quad w \wedge dw = (x\,dy + dz) \wedge dx \wedge dy$

$= dz \wedge dx \wedge dy = dx \wedge dy \wedge dz \neq 0$. (!)

\# Generalization (Pfaff systems)

Let Δ_k be a distribution of order k in M, $\dim M = n$ given by the intersection of kernels of $n-k$ 1-forms

$$\Delta_k : \begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_{n-k} = 0 \end{cases}$$

The differential form version of the Frobenius theorem states that Δ_k is integrable if and only if

$$\boxed{(\heartsuit) \qquad d\omega_j \wedge \omega_1 \wedge \dots \wedge \omega_{n-k} = 0} \qquad \forall j = 1, \dots, n-k$$

If $\dim M = n = 3$, $k = 2$, we recover the previous condition

Let us check the necessity of $(\heartsuit)$, for $n = 4$, $k = 2$, for simplicity.

If $\Delta = \Delta_2$ is integrable, let (x_1, x_2, ξ_1, ξ_2) be a local coord. system such that the integral submanifolds are given by $\xi_1 = c_1$, $\xi_2 = c_2$ (and described by coordinates x_i)

Then ω_1 is of the form smooth functions

$$\omega_1 = f_1^{(1)}(x, \xi)\, d\xi_1 + f_2^{(1)}(x, \xi)\, d\xi_2$$

(the tangent spaces being spanned, at each point, by $\frac{\partial}{\partial x^i}$, $i = 1, 2$.)

Similarly $\omega_2 = f_1^{(2)}(x, \xi)\, d\xi_1 + f_2^{(2)}(x\, \xi_2)\, d\xi_2$

Also $\quad d\omega_1 = df_1^{(1)} \wedge d\xi_1 + df_2^{(1)} \wedge d\xi_2$

$\quad\quad d\omega_2 = df_1^{(2)} \wedge d\xi_1 + df_2^{(2)} \wedge d\xi_2$

Therefore:

$$d\omega_1 \wedge \omega_1 \wedge \omega_2 = \boxed{\begin{array}{c} (df_1^{(1)} \wedge d\xi_1 + df_2^{(1)} \wedge d\xi_2) \\ \wedge \\ (f_1^{(1)} d\xi_1 + f_2^{(1)} d\xi_2) \\ \wedge \\ (f_1^{(2)} d\xi_1 + f_2^{(2)} d\xi_2) \end{array}}$$

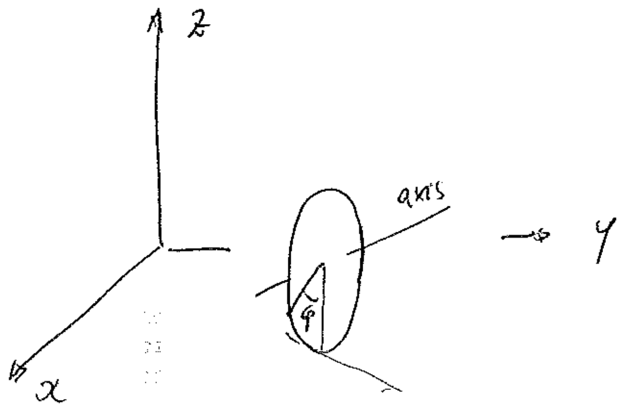
$\dots = 0$

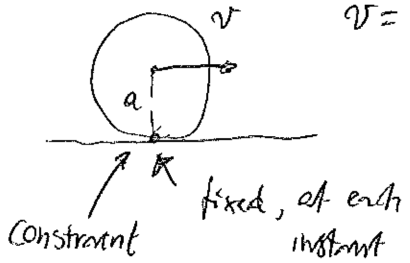
(in each summand, at least two equal differentials $d\xi_i$ occur)

Similarly $\quad d\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$

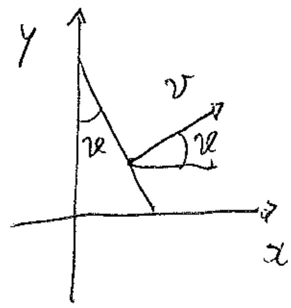
(and it is clear that the argument works in general).

* A mechanical example: a vertical disc rolling on a plane without sliding





$v = a\dot{\varphi}$



$$\dot{x} = v\cos\vartheta = a\cos\vartheta\,\dot{\varphi}$$
$$\dot{y} = v\sin\vartheta = a\sin\vartheta\,\dot{\varphi}$$

$$\begin{cases} \overbrace{dx - a\cos\vartheta\, d\varphi}^{w_1} = 0 \\ \underbrace{dy - a\sin\vartheta\, d\varphi}_{w_2} = 0 \end{cases}$$

Pfaff system

$$\begin{cases} w_1 = 0 \\ w_2 = 0 \end{cases}$$ in $\mathbb{R}^4$ (coordinates: x, y, ϑ, φ)

$$dw_1 = a\sin\vartheta\, d\vartheta\wedge d\varphi$$
$$dw_2 = -a\cos\vartheta\, d\vartheta\wedge d\varphi$$

$$w_1\wedge w_2 = dx\wedge dy - a\cos\vartheta\, d\varphi\wedge dy - a\sin\vartheta\, dx\wedge d\varphi$$

$$dw_1\wedge w_1\wedge w_2 = a\sin\vartheta\, d\vartheta\wedge d\varphi\wedge dx\wedge dy \neq 0$$
$$dw_2\wedge w_1\wedge w_2 = -a\cos\vartheta\, d\vartheta\wedge d\varphi\wedge dx\wedge dy \neq 0$$

This is an example of anholonomic constraint (not integrable).