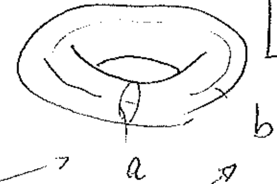
TOPOLOGIA E GEOMETRIA DIFFERENZIALE Prof. M. Spera
a.a. 2009/10

Lezione XXXVI

Omologia del toro $\mathbb{T}^2 = S^1 \times S^1$



$H_0(\mathbb{T}^2) \cong \mathbb{Z}$

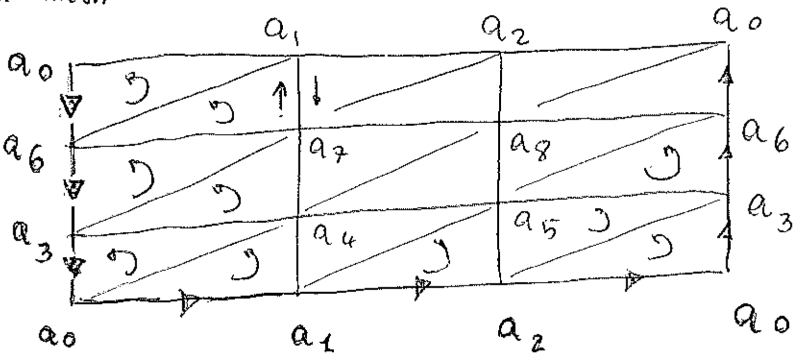
$H_1(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}$

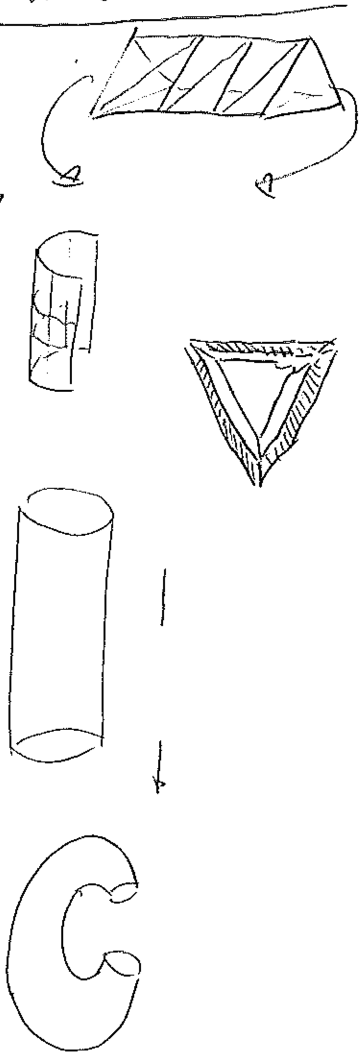
$H_2(\mathbb{T}^2) \cong \mathbb{Z}$

$c = \Sigma(\nabla + \Delta)$ 2-ciclo, non è bordo

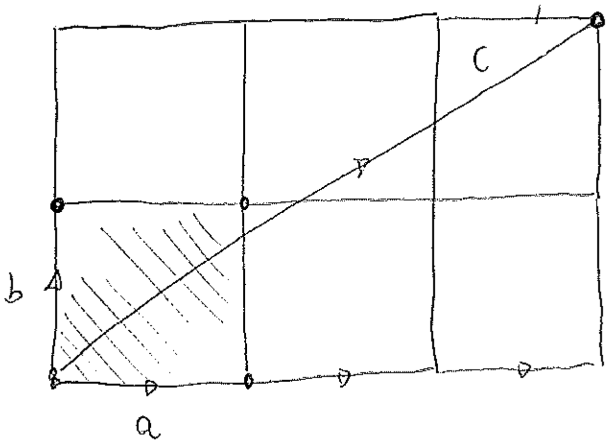
triangolazione

cf. metodo delle "mesh" in analisi numerica





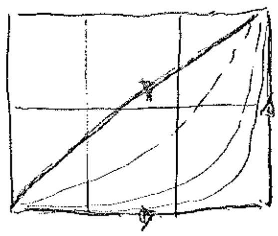
Nota: ci sono limiti inferiori al # di simplessi, v. anche oltre



$c \sim 3a + 2b$

$c \to \gamma_{(3,2)}$ nodo torico trifoglio

$\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$



$C\ \gamma_{(p,q)}$, con $(p,q) = 1$
M.C.D

i.e. p, q primi fra loro

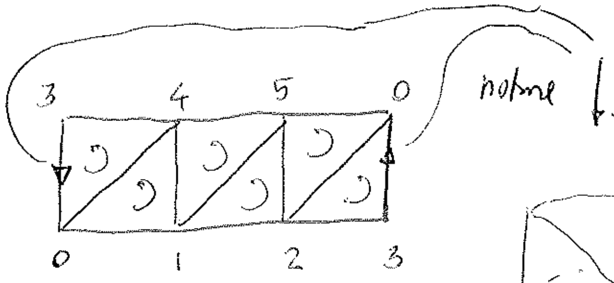
nodi torici

✧ Omologia del nastro di Möbius M

$H_0(M) = \mathbb{Z}$

$H_1(M) = \mathbb{Z}$

$H_2(M) = 0$



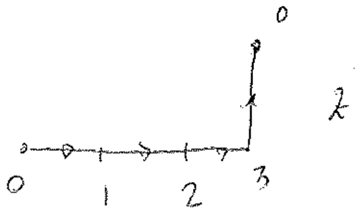
notare ⇅ non si elidono

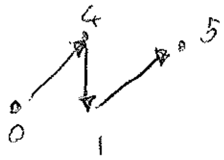
(∗)

commento: $M \sim S^1$ omotopicamente equivalente

$\boxed{H_0}$ $\quad p_0 \sim p_i \quad i = 1,2,3,4$

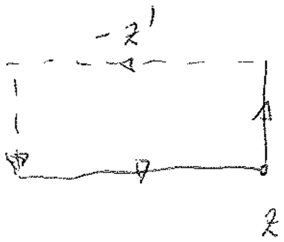
$\boxed{H_1}$



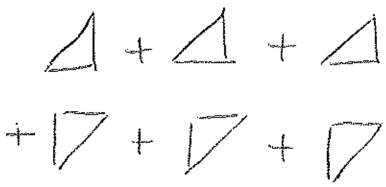


tutti gli 1-cicli non bordi sono omologhi ad un multiplo di

si elidono



$z - z' =$



con gli orientamenti stabiliti

eg: $[0,4,3] = -[0,3,4] \ldots$

(∗)

$\boxed{H_2}$ Non ci sono bordi $\quad$ Sia $C = c_0 \langle 0,3,4\rangle + c_1 \langle 0,1,4\rangle + \ldots$

In ∂C, il coefficiente di $\langle 3,4\rangle$ è c_0, e analogamente per gli altri. Pertanto, se $\partial C = 0$, è $c_0 = 0$. Sicché, se $\partial C = 0$, deve essere $c_0 = 0$.

$\Rightarrow \quad C \equiv 0$

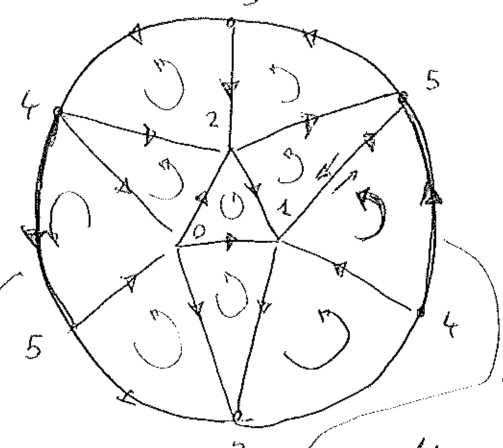
★ Omologia del piano proiettivo $\mathbb{P}^2(\mathbb{R})$

$H_0 = \mathbb{Z}$

★ commento $\pi_1(\mathbb{P}^2(\mathbb{R})) \cong \mathbb{Z}_2$, abeliano; per Hurewicz, i $H_1 \cong \mathbb{Z}_2$ direttamente

notare! → $H_1 = \mathbb{Z}_2$ ⇝ si gruppo di torsione

$H_2 = 0$ ⤷ $\mathbb{P}^2$ non è orientabile



ogni 1-simplesso è faccia di 2 2-simplessi

data un 2-ciclo w

$\partial w = 0$ implica che i coefficienti devono essere uguali

quindi

$$\partial w = 2g\,\langle a_3, a_4\rangle + 2g\,\langle a_4, a_5\rangle + 2g\,\langle a_5, a_3\rangle$$

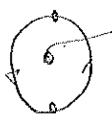
numero di incidenza = 2

$$\Rightarrow \quad g = 0$$

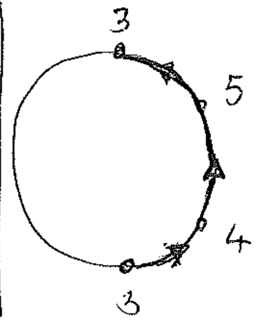
$H_1 \cong \mathbb{Z}_2$

★ ogni 1-ciclo $\sim g(\underbrace{1\,\langle a_3 a_4\rangle + 1\,\langle a_4 a_5\rangle + 1\,\langle a_5, a_3\rangle}_{z})$

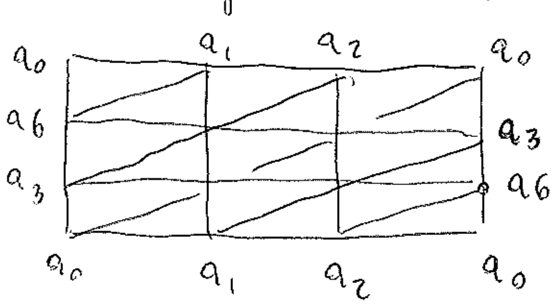
; e ogni multiplo pari di z è un bordo



$$\Rightarrow \quad H_1 \cong \mathbb{Z}_2$$



K : bottiglia di Klein

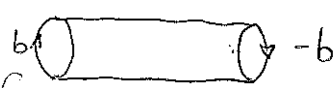


$H_0(K) \cong \mathbb{Z}$

$H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$

$H_2(K) = 0$ K non orientabile

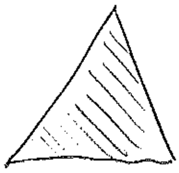
spiegazione



$b = -b \Rightarrow \boxed{2b = 0}$

inalterato

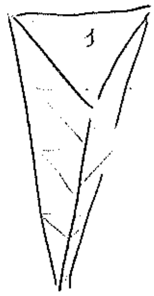
addendum 1. sull'operatore K



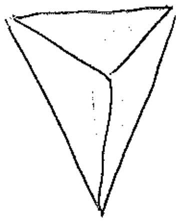
σ

$K\sigma$

"pieno"



$\partial K \sigma$

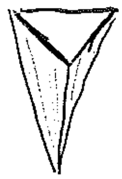


"vuoto"
(tutte le facce)

$\partial \sigma$

vuoto

$K \partial \sigma =$



tetraedro
(vuoto) privato di σ

$$\Rightarrow \qquad \partial K \sigma - K \partial \sigma = \pm \sigma$$

addendum 2. Se nell'omologia si usassero coefficienti reali (o razionali), si perderebbe la torsione

es. $H_1(K, \mathbb{R}) \cong \mathbb{R}$