Fundamentals of Signal Decompositions

"A journey of a thousand miles
must begin with a single step."

Lao-Tzu, *Tao Te Ching*

Notations

Hilbert spaces

Linear algebra

Fourier theory and sampling

Signal processing

Time-frequency representations

Prerequisites

Fourier Basics

- Fourier transform (basic properties)
- Fourier series
- discrete-time Fourier transform and series

Linear Algebra

- basics (bases, linear independence, etc)
- special matrices (Toeplitz, circulant, unitary, etc)

Basic signal processing

- sampling theorem
- convolution theorem

Review

- Hilbert spaces
- projection theorem
- multirate signal processing
- time-frequency analysis

Notations

- inner products

$$\langle f, g \rangle = \int_t f^*(t)\, g(t) dt \qquad \langle f, g \rangle = \sum_n f^*[n] g[n]$$

- roots of unity

$$W_N = e^{-j(2\pi/N)} \qquad W_N^{(k + lN)} = 1 \qquad \sum_{k=0}^{N-1} W_N^{kn} = N$$

- usual sets: $\aleph$, $\Im$, $\Re$, C
- vectors, matrices: ''bold'' letters (**v** and **M**)
- functions of continuous-time: $f(t)$ with FT $F(\omega)$
- sequences of discrete-time: $x[n]$ with DTFT $X(e^{j\omega})$
- z-transform

$$X(z) = \sum_n x[n] z^{-n}$$

Vector spaces

Def: A vector space over the set of real/complex numbers $\Re, C$ is a set of vectors E together with

- addition of vectors $x + y$, x, y in E
- scalar multiplication $\alpha\, x$, α in $\Re, C$ and x in E

Def: An inner product is a real/complex valued function <*.,.*> defined on $E \times E$

Def: An inner product space is a vector space together with an inner product

Def: A complete inner product space is a Hilbert space

- separable Hilbert spaces admit orthonormal bases

Hilbert spaces

''Infinite dimensional Euclidean spaces''...

Norms

$$\|f\| = \langle f, f \rangle^{1/2} = \left(\int_t f^*(t)f(t)dt\right)^{1/2} \qquad \|x\| = \langle x, x \rangle^{1/2} = \left(\sum_n x^*[n]x[n]\right)^{1/2}$$

Examples

- $L_2(\Re)$: space of square-integrable functions
- $l_2(\Im)$: space of square-summable sequences

Key concept: orthogonality

- **two vectors are called orthogonal if** $\langle x, y \rangle = 0$

Hilbert spaces

Basis for a space: complete set of vectors $\{\varphi_i\}$, i in $\Im$

- orthonormal case:

$$\langle \varphi_i, \varphi_j \rangle = \delta[i-j] \qquad \text{and}$$

$$f = \sum_i \langle \varphi_i, f \rangle \varphi_i \qquad \text{with} \qquad \alpha_i = \langle \varphi_i, f \rangle$$

  are called Fourier coefficients

- biorthogonal case: there is a dual basis such that

$$\langle \psi_i, \varphi_i \rangle = \delta[i-j] \qquad \text{and}$$

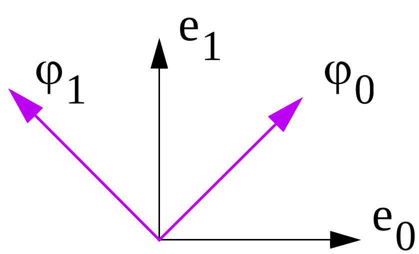
$$f = \sum_i \langle \varphi_i, f \rangle \psi_i = \sum_i \langle \psi_i, f \rangle \varphi_i \quad \text{with } \alpha_i = \langle \varphi_i, f \rangle, \ \beta_i = \langle \psi_i, f \rangle,$$

- overcomplete: vectors are complete
  but linearly dependent, frames, tight frames

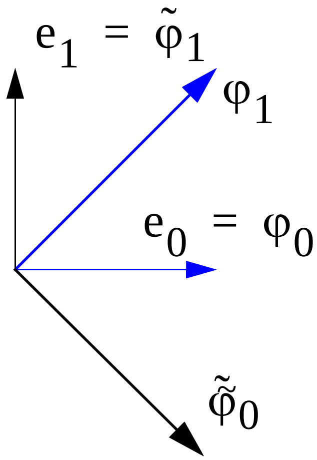
Hilbert spaces

Example: $\Re^2$

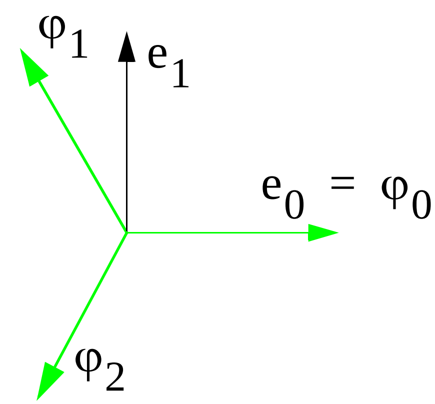
orthogonal basis



biorthogonal basis



tight frame



Note: quantization in orthogonal case is easy, unlike in the other cases

Hilbert spaces ...
... orthonormal bases

Theorem: Given an orthonormal system $\{x_1, x_2, \ldots\}$ in a Hilbert space H, the following are equivalent:

- The set $\{x_1, x_2, \ldots\}$ is an orthonormal basis for H.
- If $\langle x_i, y\rangle = 0$ for $i = 1, 2, \ldots$ then $y = 0$.
- The span of $\{x_1, x_2, \ldots\}$ is dense in H, that is, every vector in H is a limit of a sequence of vectors in the span of $\{x_1, x_2, \ldots\}$.
- Parseval's equality: For every y in H,

$$\|y\|^2 = \sum_i \left|\langle x_i, y\rangle\right|^2.$$

- Generalized Parseval's equality: For every y_1, y_2 in H,

$$\langle y_1, y_2\rangle = \sum_i \langle x_i, y_1\rangle^* \langle x_i, y_2\rangle.$$

Hilbert spaces ...
... orthogonal projections

Given a Hilbert space H and a closed subspace S:

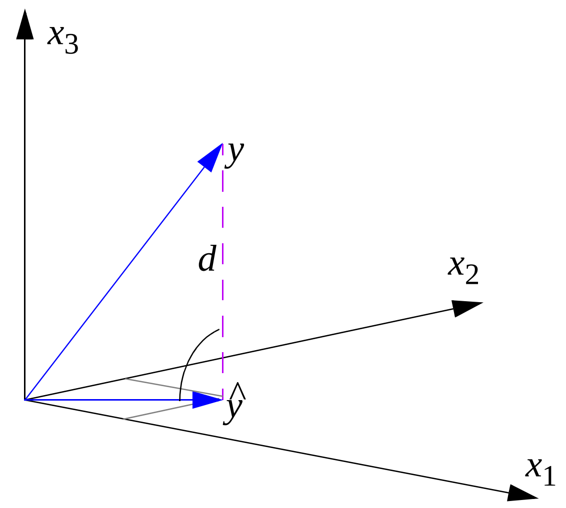
$$\mathrm{H} = \mathrm{S} \oplus \mathrm{S}^{\perp},$$

where $\mathrm{S}^{\perp}$ is the orthogonal complement of S in H.

Moreover, if $u \in \mathrm{H}$ then

$$u = v + w$$

where $v \in \mathrm{S}$ and $w \in \mathrm{S}^{\perp}$.



Def: An operator P is called a projection operator onto S if

$$P(v + w) = v$$

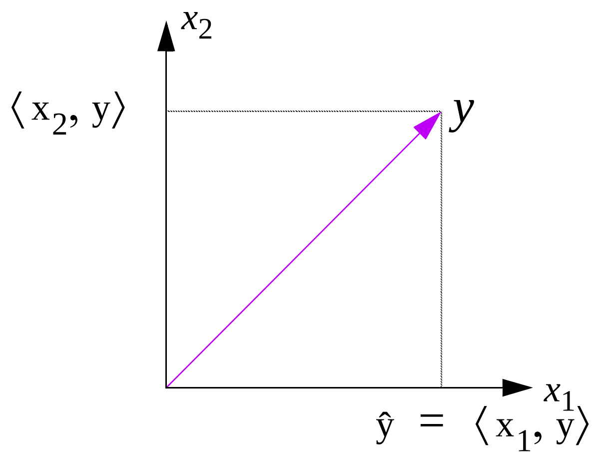
Result: An operator P is a projection operator if and only if it is

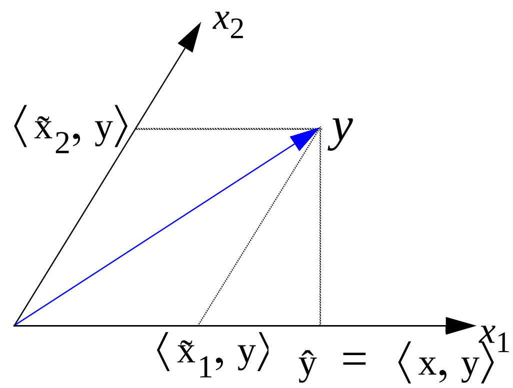
- idempotent: $P^2 = P$, and
- self-adjoint: $P^* = P$.

Hilbert spaces ...
... orthogonal projections

Orthogonal projection

- best subspace approximation is an orthogonal projection
- error is orthogonal to approximation





Successive approximation

- orthonormal bases successively approximate (left)
- biorthogonal bases and frames do not (right)

Linear algebra

Finite dimensional Hilbert spaces $\Re^n$**,** C^n

Unitary matrices: U **such that** $U^* U = I$

Special matrices such as DFT matrix $F_{ij} = W_N^{ij}$

Convolution Theorem: $F^* C F = L$

Matrices of polynomials or rational functions $H(z)$

- MIMO LTI's
- normal rank: maximum rank of $H(z)$
- unitary matrices of polynomials (on unit circle): paraunitary

Sampling theorem

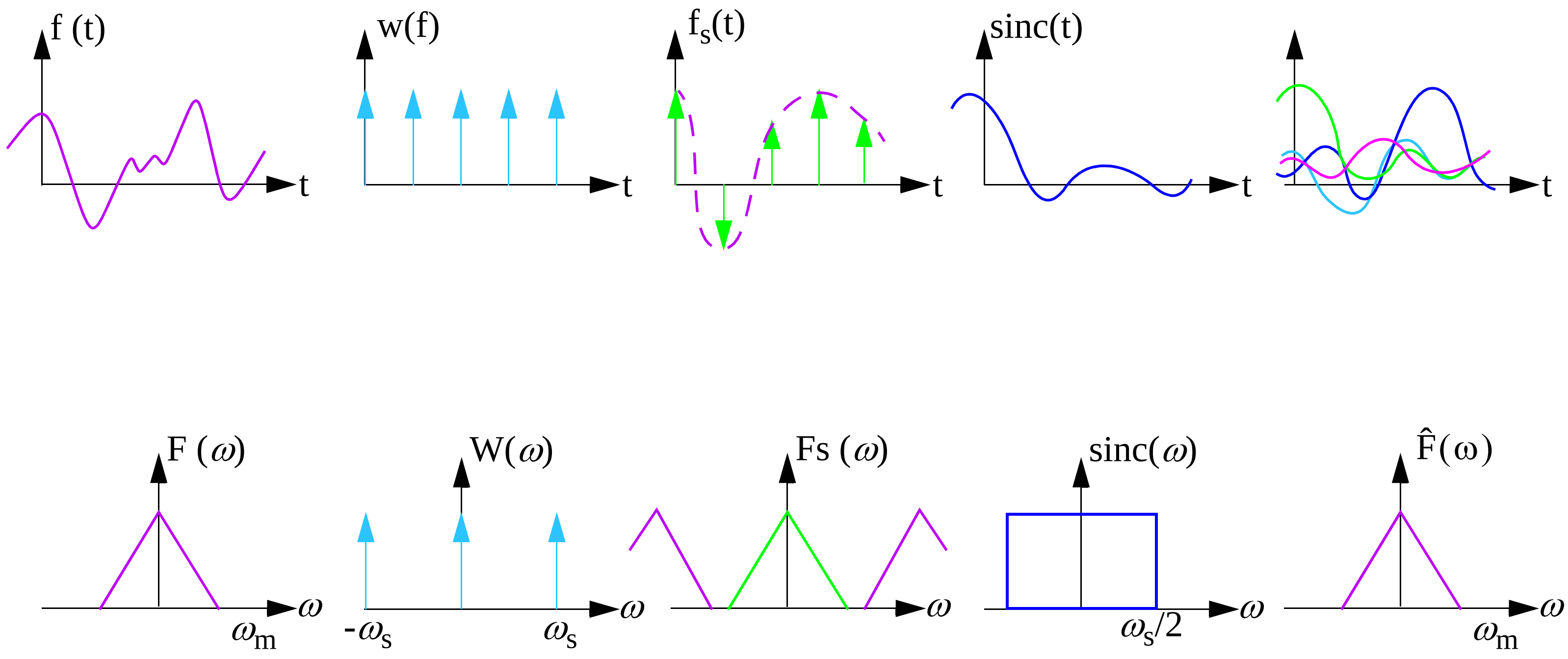
Theorem: If $f(t)$ is continuous and bandlimited to ω_m, then $f(t)$ is uniquely defined by its samples taken at twice ω_m or $f(n\pi/\omega_m)$. The minimum sampling frequency is $\omega_s = 2\omega_m$ and $T = \pi/\omega_m$ is the maximum sampling period. Then $f(t)$ can be recovered by:

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\frac{\sin(\pi t/T)}{\pi t/T}$$

Another view: $\mathrm{sinc}(t-l)$, l in $\Im$ **, is an orthonormal basis for functions bandlimited to** $(-\pi,\pi)$, **where** $\mathrm{sinc}(t)$ **is**

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Sampling theorem...
... pictorially



Expansions

Continuous-time integral transform or expansion

$$x(t) = \int X(\omega) \cdot \varphi_{\omega}(t) d\omega \qquad X(\omega) = \langle \tilde{\varphi}_{\omega}(t), x(t) \rangle$$

Example: Fourier transform

$$\varphi_{\omega}(t) = \frac{1}{2\pi} \cdot e^{j\omega t} \qquad \tilde{\varphi}_{\omega}(t) = e^{j\omega t}$$

Continuous-time series expansion

$$x(t) = \sum X[i] \cdot \varphi_i(t) \qquad X[i] = \langle \tilde{\varphi}_i(t), x(t) \rangle$$

Example: Fourier series [0,1]

$$\varphi_{\omega}(t) = e^{j2\pi kt} \qquad \tilde{\varphi}_{\omega}(t) = e^{j2\pi kt}$$

Expansions

Discrete-time integral transform or expansion

$$x[n] = \int X(\omega) \cdot \varphi_{\omega}[n] d\omega \qquad X(\omega) = \langle \tilde{\varphi}_{\omega}[n], x[n] \rangle$$

Example: Discrete-time Fourier transform

$$\varphi_{\omega}[n] = \frac{1}{2\pi} \cdot e^{j\omega n} \qquad \tilde{\varphi}_{\omega}[n] = e^{j\omega n}$$

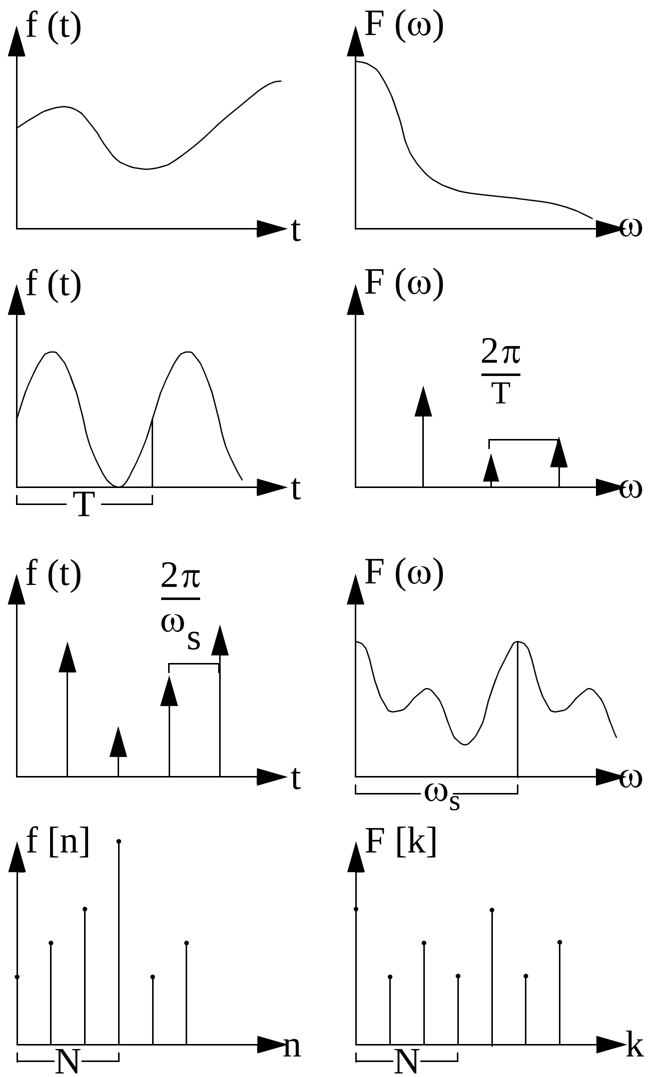
Discrete-time series expansion

$$x[n] = \sum X[i] \cdot \varphi_i[n] \qquad X[i] = \langle \tilde{\varphi}_i[n], x[n] \rangle$$

Example: Discrete-time Fourier series [0,N-1] (or DFT)

$$\varphi_k[n] = \frac{1}{N} \cdot e^{(j2\pi kn)/N} \qquad \tilde{\varphi}_k[n] = e^{(j2\pi kn)/N}$$

Fourier theory



CTFT

CTFS

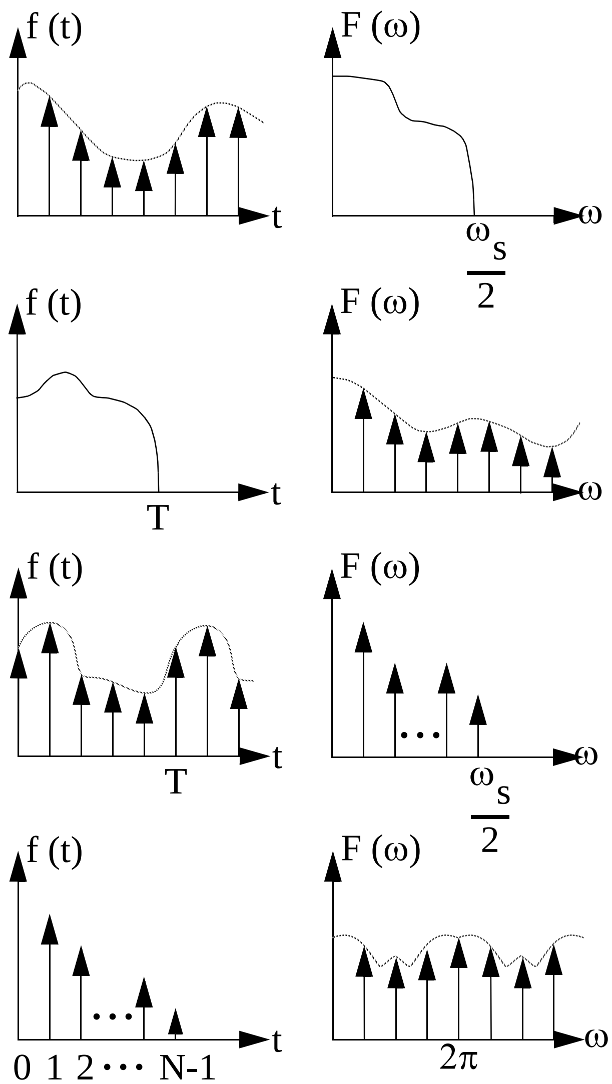
DTFT

DTFS

Fourier theory

Transform	Time	Freq.	Analysis Synthesis	Duality
Fourier transform CTFT	C	C	$F(\omega) = \int_t f(t) e^{-j\omega t} dt$ $f(t) = \frac{1}{2\pi} \int_\omega F(\omega) e^{j\omega t} d\omega$	self-dual
Fourier series CTFS	C P	D	$F[k] = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-j2\pi k \frac{t}{T}} dt$ $f(t) = \sum_k F[k] e^{j2\pi k \frac{t}{T}}$	dual with DTFT
Discrete-time Fourier transform DTFT	D	C P	$F(e^{j\omega}) = \sum_n f[n] e^{-j2\pi n \frac{\omega}{\omega_s}}$ $f[n] = \frac{1}{\omega_s} \int_{-\frac{\omega_s}{2}}^{\frac{\omega_s}{2}} F(e^{j\omega}) e^{j2\pi n \frac{\omega}{\omega_s}} d\omega_s$	dual with CTFS
Discrete-time Fourier series DTFS	D P	D P	$F[k] = \sum_{n=0}^{N-1} f[n] e^{-j2\pi k \frac{n}{N}}$ $f[n] = \sum_{k=0}^{N-1} F[k] e^{-j2\pi k \frac{n}{N}}$	self-dual

Fourier theory



BL-CTFT

FL-CTFT

BL-CTFS

FL-DTFT

Fourier theory

Transform	Time	Frequency	Equivalence	Duality
Fourier transform of bandlimited signal BL-CTFT	Can be sampled	$\left(-\frac{\omega_s}{2}, \frac{\omega_s}{2}\right)$	Sample time. Periodize frequency.	dual with FL-CTFT
Fourier transform of finite-length signal FL-CTFT	$(0, T)$	Can be sampled	Periodize time. Sample frequency.	dual with BL-CTFT
Fourier series of bandlimited periodic signal BL-CTFS	Periodic can be sampled	Finite number of Fourier coefficients	Sample time. Finite Fourier series in time.	dual with FL-DTFT
Discrete-time Fourier transform of finite-length sequence FL-DTFT	Finite number of samples	Periodic can be sampled	Sample frequency. Finite Fourier series in frequency.	dual with BL-CTFS

Multirate Digital Signal Processing

Branch of DSP

- not a single, but multiple sampling rates
- important in many applications
- analysis more complex

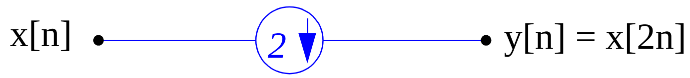
Key points

- aliasing!
- complex exponentials are not eigenfunctions anymore
- linear periodically time-varying (LPTV) systems

Key analysis methods

- map single input single output (SISO) LPTV system into a multiple input multiple output (MIMO) linear time-invariant system

Multirate DSP...
... downsampling by 2



$$\ldots x[0]x[1]x[2]\ldots \quad \ldots x[0]x[2]x[4]\ldots$$

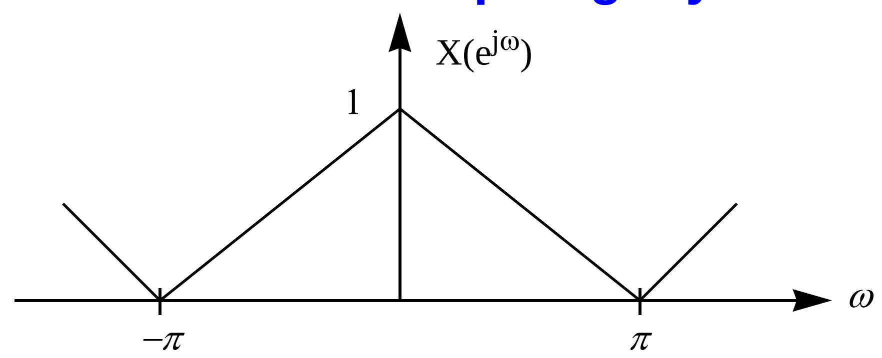
- $y = D_2 \cdot x$

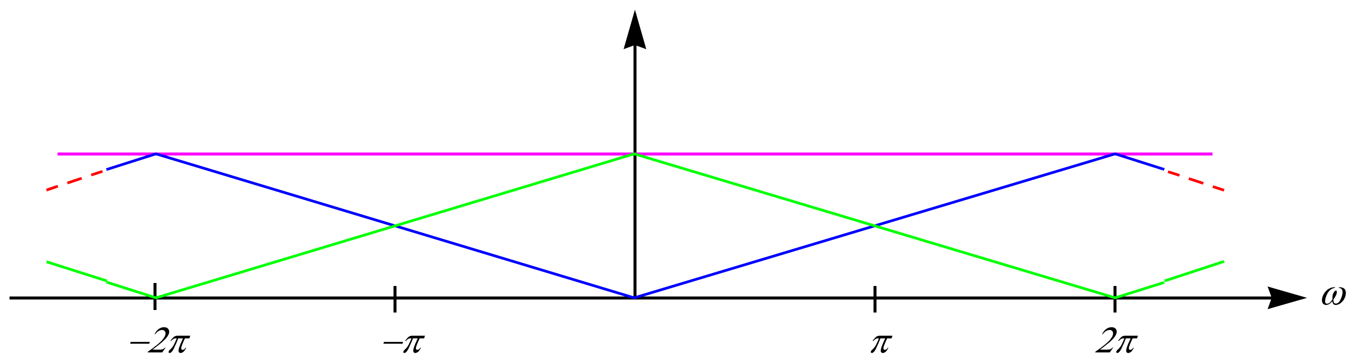
$$\begin{bmatrix} \ldots \\ y[-1] \\ y[0] \\ y[1] \\ \ldots \end{bmatrix} = \begin{bmatrix} . & . & . & . & . & . \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ . & . & . & . & . & . \end{bmatrix} \cdot \begin{bmatrix} \ldots \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ \ldots \end{bmatrix}$$

$$Y(z) = \frac{1}{2}[X(z^{1/2}) + X(-z^{1/2})]$$

$$Y(e^{j\omega}) = \frac{1}{2}[X(e^{j\omega/2}) + X(e^{j(\omega - 2\pi)/2})]$$

Multirate DSP...
... downsampling by 2



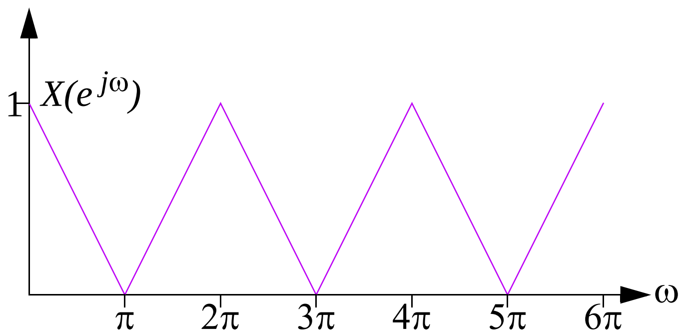


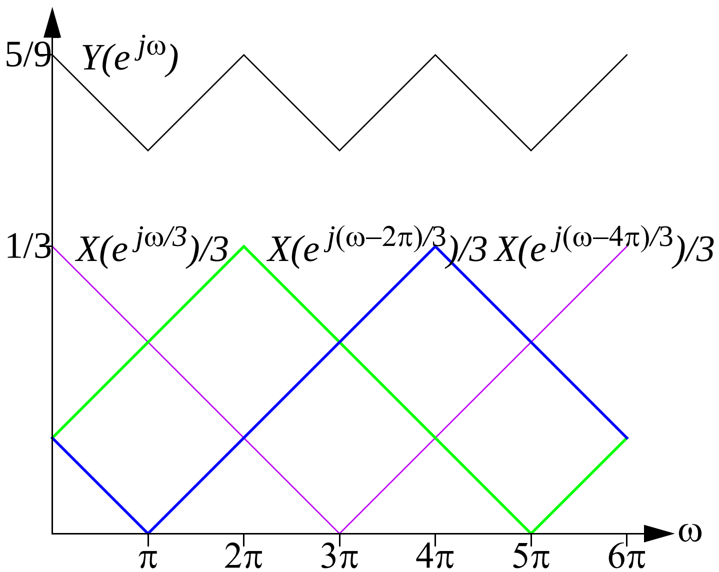
Note: downsampling = subsampling ~ decimation

Multirate DSP...
... downsampling by N

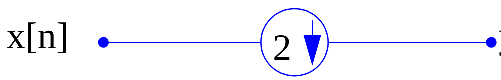
Downsampling by N **in time domain:** …x[0],x[1],x[2]… **, that is,** …x[0],x[N],x[2N]… **, produces in frequency domain:**

$$Y(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega - k2\pi)/N})$$



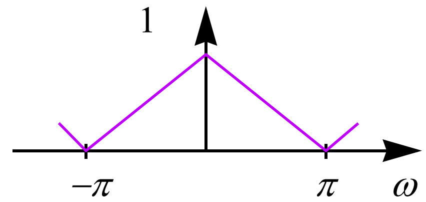


Multirate DSP...
... upsampling by 2

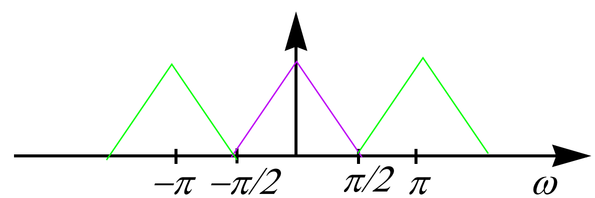


$$y[n] = \begin{cases} x[n/2] & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$\ldots x[0]x[1]x[2]\ldots$ $\quad\quad$ $\ldots x[0]\ 0\ x[1]\ 0\ x[2]\ \ldots$

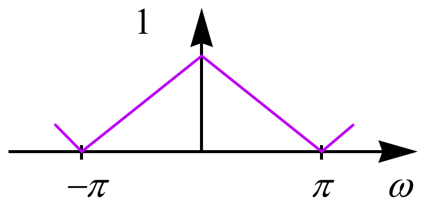


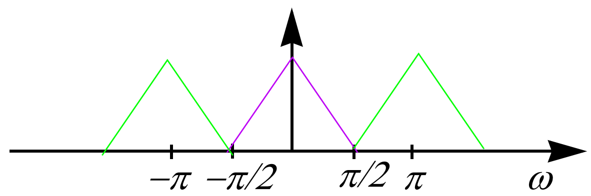
- $y = U_2 x$
- $$\begin{bmatrix} \ldots \\ y[0] \\ y[1] \\ \ldots \end{bmatrix} = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 1 & 0 & 0 & \ldots \\ \ldots & 0 & 0 & 0 & \ldots \\ \ldots & 0 & 1 & 0 & \ldots \\ \ldots & 0 & 0 & 0 & \ldots \\ \ldots & 0 & 0 & 1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} \begin{bmatrix} \ldots \\ x[0] \\ x[1] \\ x[2] \\ \ldots \end{bmatrix}$$



- $$Y(z) = X(z^2)$$
  $$Y(e^{j\omega}) = X(e^{j2\omega})$$

Multirate DSP...
... upsampling by 2



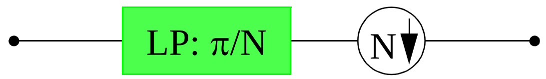


Upsampling by N

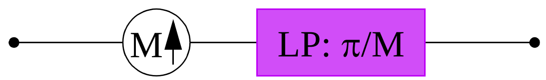
$$Y(z) = X(z^N) \qquad Y(e^{j\omega}) = X(e^{jN\omega})$$

Multirate DSP...
... basic operations

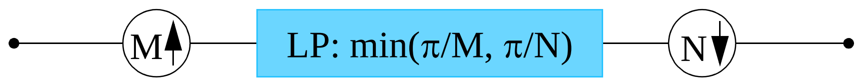
Filtering before downsampling



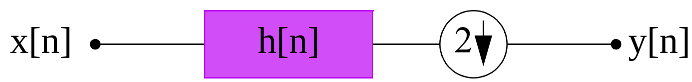
Filtering after upsampling



Rational sampling rate change



Basic operations...
... filtering before downsampling



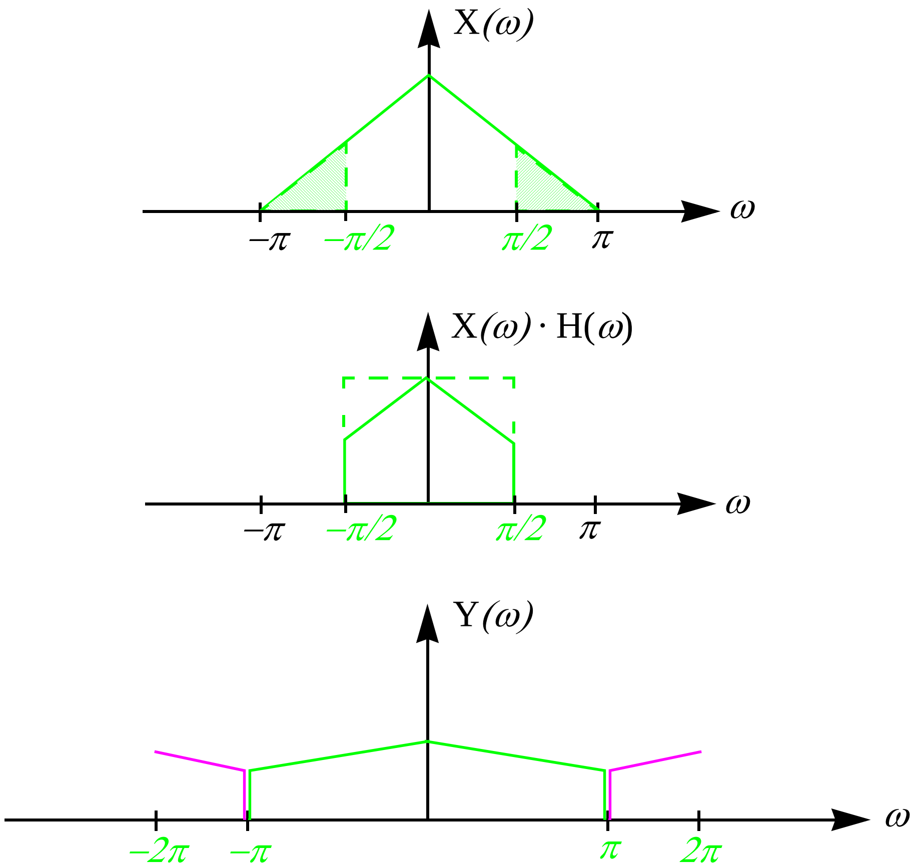
- $$H = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & h[2] & h[1] & h[0] & 0 & 0 & \ldots \\ \ldots & h[3] & h[2] & h[1] & h[0] & 0 & \ldots \\ \ldots & h[4] & h[3] & h[2] & h[1] & h[0] & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$ Toeplitz

- $y = D_2 \cdot H \cdot x$

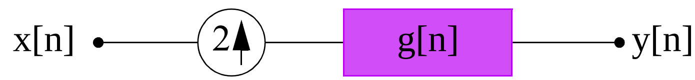
$$y = \begin{bmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & h[2] & h[1] & h[0] & 0 & 0 & 0 & 0 & \ldots \\ \ldots & h[4] & h[3] & h[2] & h[1] & h[0] & 0 & 0 & \ldots \\ \ldots & h[6] & h[5] & h[4] & h[3] & h[2] & h[1] & h[0] & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix} x$$

Basic operations...
... filtering before downsampling

$$Y(z) = \frac{1}{2}[H(z^{1/2})X(z^{1/2}) + H(-z^{1/2})X(-z^{1/2})]$$



Basic operations...
... filtering after upsampling



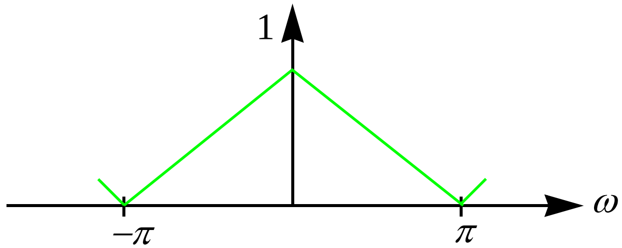
- $$G = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & g[2] & g[1] & g[0] & 0 & 0 & \cdots \\ \cdots & g[3] & g[2] & g[1] & g[0] & 0 & \cdots \\ \cdots & g[4] & g[3] & g[2] & g[1] & g[0] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$ Toeplitz

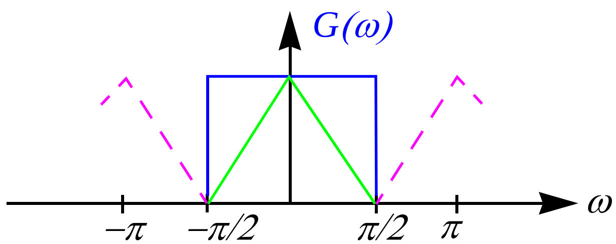
- $y = G \cdot U_2 \cdot x$

$$y = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & g[0] & 0 & 0 & \cdots \\ \cdots & g[1] & 0 & 0 & \cdots \\ \cdots & g[2] & g[0] & 0 & \cdots \\ \cdots & g[3] & g[1] & 0 & \cdots \\ \cdots & g[4] & g[2] & g[0] & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} x$$

Basic operations...
... filtering after upsampling

- $Y(z) = G(z) \cdot X(z^2)$

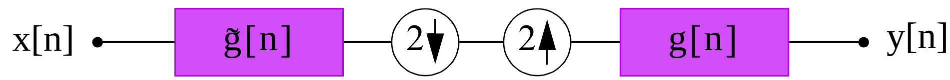




- If $g[n] = h[-n]$, then

$$G \cdot U_2 = (D_2 H)^T$$

Basic operations...
... smoothing operator



where $\langle g[n], g[n-2k] \rangle = \delta_k$ and $\tilde{g}[n] = g[-n]$ (*)

Claim: The operator P mapping $x \to y$ is a projection

Proof:

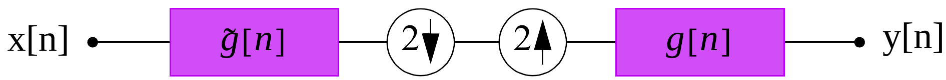
$$P = G \cdot U_2 \cdot D_2 \cdot \tilde{G}$$

Since $\tilde{G} = G^T$ and $D_2\tilde{G} \cdot GU_2 = I$ because of (*) then

$$P^2 = G \cdot U_2 \cdot D_2 \cdot \tilde{G} \cdot G \cdot U_2 \cdot D_2 \cdot \tilde{G} = GU_2 \cdot I \cdot D_2 \cdot \tilde{G} = P$$

$$P^* = (G \cdot U_2 \cdot D_2 \cdot \tilde{G})^T = G(U_2 \cdot D_2)^T \cdot \tilde{G} = G \cdot U_2 \cdot D_2 \cdot \tilde{G} = P$$

Basic operations...
... piecewise constant approximation



with $g[n] = \frac{1}{\sqrt{2}}[\delta[n] + \delta[n-1]]$ and $\|g\|_2 = 1$

Since $D_2 \cdot \tilde{G} = \frac{1}{\sqrt{2}}\begin{bmatrix} \dots & 1 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 1 & \dots \\ \dots & 0 & 0 & 0 & 0 & \dots \end{bmatrix}$ and $G \cdot U_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} \dots & 1 & 0 & \dots \\ \dots & 1 & 0 & \dots \\ \dots & 0 & 1 & \dots \\ \dots & 0 & 1 & \dots \end{bmatrix}$

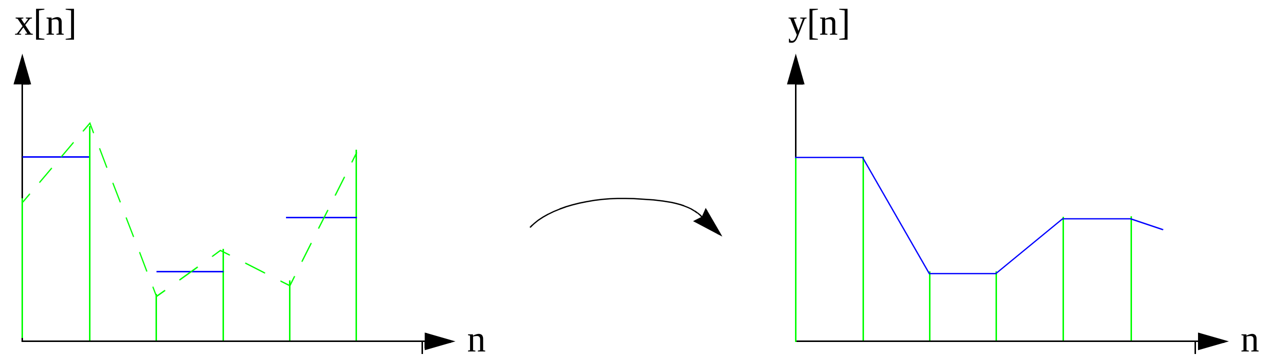
then

$$P = G \cdot U_2 D_2 \cdot \tilde{G} = \frac{1}{2}\begin{bmatrix} \dots & 1 & 0 & \dots \\ \dots & 1 & 0 & \dots \\ \dots & 0 & 1 & \dots \\ \dots & 0 & 1 & \dots \end{bmatrix}\begin{bmatrix} \dots & 1 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 1 & \dots \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \dots & 1 & 1 & 0 & 0 & \dots \\ \dots & 1 & 1 & 0 & 0 & \dots \\ \dots & 0 & 0 & 1 & 1 & \dots \\ \dots & 0 & 0 & 1 & 1 & \dots \end{bmatrix}$$

Basic operations...
... piecewise constant approximation

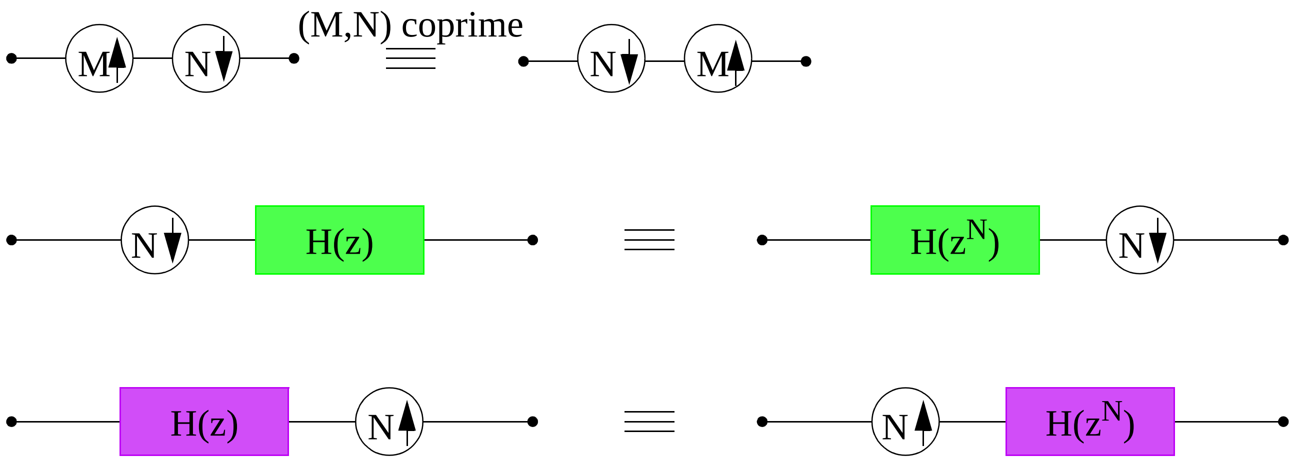
Therefore

$$\{x[0]x[1]x[2]\ldots\} \to \left\{\frac{1}{2}(x[0]+x[1]), \frac{1}{2}(x[0]+x[1]), \ldots\right\}$$

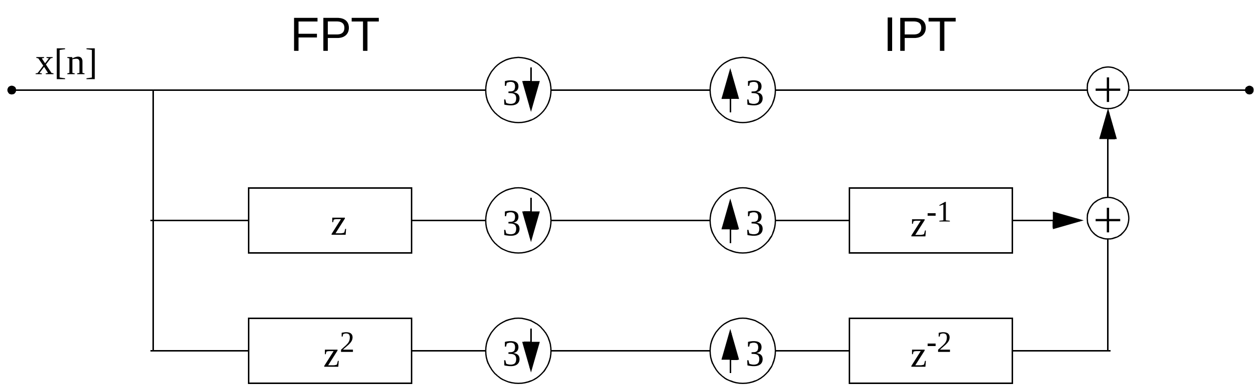


Multirate DSP

Multirate identities



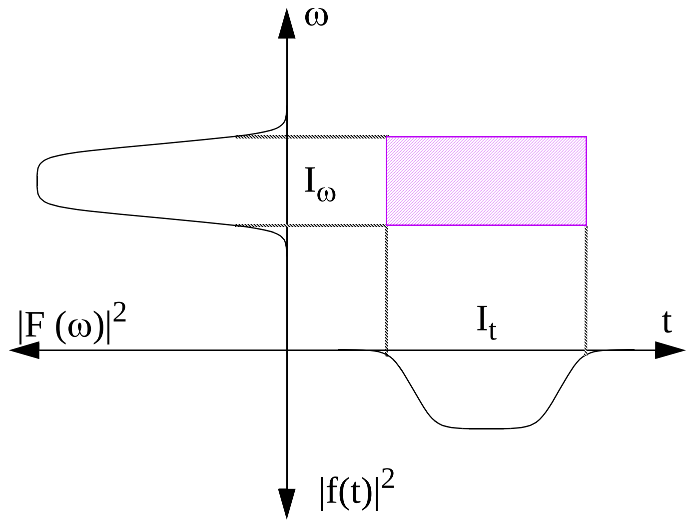
Polyphase transform



Time-frequency representations

Basis functions have some spread in time and frequency

- leads to time-frequency tile or atom



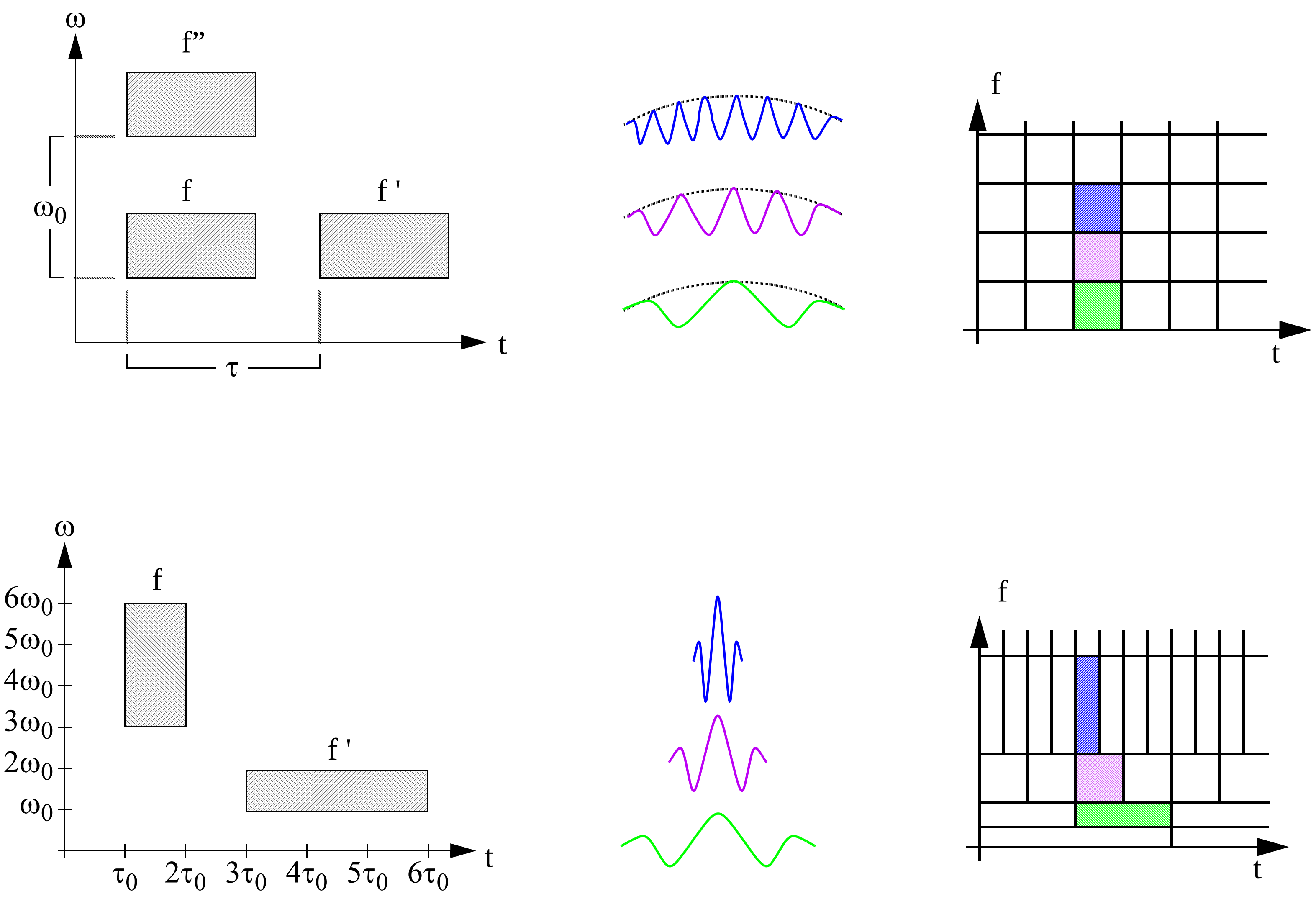
Joint time-frequency resolution is lower bounded by uncertainty principle:

$$\Delta_\tau^2 \cdot \Delta_\omega^2 \geq \frac{\pi}{2}$$

where Δ_τ^2 **and** Δ_ω^2 **are the integrals of** $t^2|f(t)|^2$ **and** $\omega^2|F(\omega)|^2$

Can trade time for frequency resolution and vice-versa

STFT versus WT



Scale and resolution

Scale is as in maps:

- large scale ⇔ less details, large area
- small scale ⇔ detail, small area

Resolution: ~ information