University of Verona
Master's Program in Mathematics
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Partial test in Optimization

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Solve one between Exercises 1 and Exercise 2, and solve obligatorily Exercise 3.

Exercise 1. Let Ω be an open bounded subset of $\mathbb{R}^2$. Consider the problem:

$$\inf_{u\in H_0^1(\Omega)} \int_\Omega \Big(3|\partial_{x_1} u(x_1,x_2)|^2 + |\partial_{x_2} u(x_1,x_2)|^2 + \partial_{x_1} u(x_1,x_2)\cdot \partial_{x_2} u(x_1,x_2) + \\ + \left((4x_2+1)\,u(x_1,x_2) - x_1\right)^2\Big)\, dx_1\, dx_2.$$

(1) Prove that the problem admits a unique solution.
(2) Formulate the problem in the form $\mathscr{F}(u) = F(u) + G\circ\Lambda(u)$, where $F : X \to]-\infty,+\infty]$, $G : Y \to]-\infty,+\infty]$, and $\Lambda : X \to Y$, carefully precising the functional spaces X, Y and discuting the regularity of F, G, Λ.
(3) Write the dual problem and the extremality relations. Establish if the dual problem admits an unique solution.
(4) Using the previous results, write down a partial differential equation satisfied by the minimum.

Exercise 2. Let Ω be an open bounded subset of $\mathbb{R}^d$, $q\in H^1(\Omega;\mathbb{R}^d)$ be fixed. Set:

$$\mathscr{C} := \{v\in H_0^1(\Omega;\mathbb{R}) : \|\nabla v - \nabla q\|_{L^2(\Omega;\mathbb{R}^d)} \leq 1\}.$$

Consider the problem problema

$$\inf_{u\in\mathscr{C}} \int_\Omega \frac{|u(x)|^2}{2}\, dx.$$

(1) Prove that the problem admits a unique solution.
(2) Formulate the problem in the whole space in the form $\mathscr{F}(u) = F(u) + G\circ\Lambda(u)$, where $F : X \to]-\infty,+\infty]$, $G : Y \to]-\infty,+\infty]$, and $\Lambda : X\to Y$, carefully precising the functional spaces X, Y and discuting the regularity of F, G, Λ.
(3) Write the dual problem and the extremality relations. Establish if the dual problem admits an unique solution.

Exercise 3.

(1) Let $f_i : \mathbb{R}\to\mathbb{R}$ be continuous functions, $i = 1,\dots,N$. Define $F : \mathbb{R} \rightrightarrows \mathbb{R}$ by setting

$$F(x) = \Big[\min\{f_i(x) : i = 1,\dots,N\}, \max\{f_i(x) : i = 1,\dots,N\}\Big].$$

Establish if F is a continuous set-valued map by exhibiting a proof in the affirmative case or a counterexample in the negative case.

(2) Let $p : \mathbb{R}^d \to \mathbb{R}$ be a function satisfying $p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ for every $x\in\mathbb{R}^d$ and $\lambda \geq 0$. Prove that there exists a closed convex set $C \subseteq \mathbb{R}^d$ such that $p(x) = \sigma_C(x)$.

(3) Prove that the distance function from a closed convex subset of a normed space is a convex function.

(4) Given $A = \{(x,y)\in\mathbb{R}^2 : |x+y| < 2 \text{ and } |x-y| < 2\}$ compute its polar A^0.

(5) Define $f : \mathbb{R}^2\to\mathbb{R}$ by setting $f(x,y) = e^{\sqrt{x^2+y^2}}$. Compute its convex conjugate f^*.