MORSE THEORY

Contents:

0 - Main idea: Use functions to get knowledge of the topology of the manifold

1 - Attaching cells & CW complexes (alteration of homology)
   (homology, betti numbers)
   Ex: circle, sphere, torus (I) ∞

2 - Theorem (Milnor) "Every M (compact) is a CW" (statement)

3 - Critical points, tubeshape, Morse lemma (Critical points are isolated) functions + attaching cells at every critical value of f (ex: circle)

4 - Theorems
   1. Retraction \( a \rightarrow b \)
   2. Attaching cells \( a \rightarrow b \)

5 - Existence of (Morse functions) by sand (ex M)

4 - Milnor's Theorem 

5 - Applications
   1. \( P \times N \)
   2. Morse inequalities
   3. Reidemeister's formula (Milnor spheres)

TOPOLOGIA E GEOMETRIA DIFFERENZIATE

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Lazione A6
1 - Attaching cells

\[ e^m = \bigvee_{n=0}^{m-1} S^n \]

\[ f: S^{m-1} \times \mathbb{R} \to f(x) \]

\[ \times U e^n = \times \bigvee_{n=0}^{m-1} e^n / f(x) n u \]

\[ CW \text{ complex} \quad (C \text{ closed if } C_n \text{ cell close } \Rightarrow \text{ weak topology}) \]

\[ \chi = \sum (-1)^k b_k = \sum (-1)^k c_k \]

\[ \begin{array}{c}
\text{Betti} \\
\text{independent cycles} \leftrightarrow \text{independent closed forms} \\
\text{d}c = 0 \quad \langle c, w \rangle = \int c w \\
\text{Stokes:} \quad \langle \text{d}c, w \rangle = \langle c, \text{d}w \rangle \\
\end{array} \]

\[ C = \text{Holge (harmonic forms)} \]

Examples:

- Circle
- Sphere
- Torus

2 - Milnor's Theorem (Statement)
Critical point: \( df(p) = 0 \quad \frac{\partial f}{\partial x^i}(p) = 0 \quad \forall i \)

Monodeficiency: \( \det H(p) = \det \left( \frac{\partial^2 f}{\partial x^i \partial x^j} \right) \neq 0 \)

\( \Rightarrow \) \( n \cdot \lambda > 0 \) (change of coordinates)

Jacobian:
\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x^i} & \frac{\partial f}{\partial y^i}
\end{pmatrix}
\]

Index = \( n^2 \) of negative eigenvalues

\( \Rightarrow \) relevant \( \lambda \) (algebra or partition)

Morse function: Morse lemma. You can choose coordinates in \( p \) which fit the
\[
f = f(p) - x_1^2 - x_2^2 - \ldots - x_k^2 + x_{k+1}^2 + \ldots + x_n^2.
\]

\( \Rightarrow \) critical (non-degenerate) one isolated \( \lambda \) and finite
THE TORUS

$f = \text{height}$

$\{ f \leq c(y) \}$

Sublevels

**MILNOR'S PROOF**

$f$ Morse

$1 - M_a = f^{-1}[-\infty, a]$  

Reduction

If $[-c, b]$ compact and does not contain critical points

Then $M_a \cap M_b$

**Proof** choose $\langle \cdot, \cdot \rangle$ generic. $\langle df(Y), Y \rangle = df(Y)$

$x = -\frac{df}{\|df\|}$ makes sense
2. Attaching cells

\[ f([a, b]) \text{ compact } \exists \rho \text{ s.t. } \forall x, y \in \mathbb{R} \] \n\[ |x - y| < \rho \implies f(x) = f(y) \]

\[ M_0 \cap M_0 = M \cup e^k \]

See Figure: \( R = 1 \) \( n = 2 \) Use Morse Lemma

\[ G = \{ f \leq c + \epsilon, c \leq x \leq c + \epsilon \} \]

\[ c = c + \epsilon = c + x_1^2 + x_2^2 \]

\[ \epsilon = -x_1^2 + x_2^2 \]

\[ G \cap e^k \]

\[ \omega' = (1 + \rho) \omega \]

\[ n' = (1 - \rho) n \]

\[ \rho = \max \left( \frac{\omega'}{\omega}, \frac{n - n}{n} \right) \]

\[ \rho = \frac{n - n}{n} \]

\[ \rho = |n'| \]
There exist more functions 
embedding in Euclidean space + Sand 

6) Minor: case of proof

\( M \rightarrow \text{Whitney} \rightarrow M \otimes K \rightarrow S \) of 8

and one (sol. Ma. compact, see first)

\( Pr \rightarrow Pr \rightarrow \text{attaching process} \)

finite II index o: \( a \) cell for a given \( R \) (k: rules compare torsus)
5. Applications

1) \( \mathbb{CP}^n \) as \( \mathbb{E}^N \)

\[ (z_0: z_1: \ldots: z_n) \]

\[ f(z_0: z_1: \ldots: z_n) = \sum c_i |z_i|^2 \quad \text{will differ} \]

\[ \forall i : z_i \neq 0 \]

\[ \frac{|z_0|^2}{|z_i|^2} = x_i + iy_i \]

\[ f = c_0 |z_0|^2 + \sum c_i |z_i|^2 = c_0 |z_0|^2 + \sum c_i (x_i^2 + y_i^2) \]

\[ = c_0 \left( 1 - \sum x_i^2 + y_i^2 \right) + \sum c_i (x_i^2 + y_i^2) \]

\[ \Rightarrow P_0, P_1, P_n \quad \text{mark} \quad \begin{cases} \quad \text{mon Targets} \quad E_j < c_0 \\ \text{PK} \end{cases} \]

\[ \Rightarrow n \in \mathbb{N} \quad \mathbb{C}P^N \Rightarrow \mathbb{C}U^N \Rightarrow \mathbb{C}U^N \]

\[ H_i (\mathbb{C}P^N, \mathbb{Z}) = \sum_{i=0}^{\infty} \]

2 - Morse inequalities
\[
\begin{align*}
\Sigma (c - 1)x_2 &= \chi(M) \\
C R &> P R
\end{align*}
\]

3 - Reeb

\[ M \Rightarrow M \times S^n \]

For any two n-dim cut points

\[ P_R = \max \rightarrow 2 = E \]
\[ P_0 = \max \rightarrow 0 = \xi \]

\( M \) is obtained by attaching a 2-cell on a 0-cell of \( S \) (or 2 2-cells along their common boundary)

Nuclear sphere:

The spheres obtained in this way may not be di-homeomorphic!

(ex: on \( S^2 \) one gets \( 2^8 \) different co-orientable structures)
Picture: \( R = \sqrt{z} \quad n = 0 \)

\[ f = c + t \]

\[ f = c + t = c - x^2 + y^2 \]

"no critical points"

\[ \mathcal{M}_e + e \sim \mathcal{M}_e + e \]

If one wants to describe the

\[ \rho = \frac{v_0 - |v_1|}{|a|} \]

\[ \rho = |v_1| \]
There exist Morse functions (sketch)

\[ f_{a}(x) = f(x) + a \cdot x_{1} \cdot x_{2} + \ldots + a \cdot x_{n} \]

\[ q(x) = \left( \begin{array}{c}
\frac{2f_{1}}{2x_{1}} \\
\vdots \\
\frac{2f_{n}}{2x_{n}}
\end{array} \right) \]

\[ H(f) = J(q) \]

Consider for \( f \to 0 \) \( q(x) = 0 \) and \( J(q) \) nonsingular

\[ g_{a}(x) = g(x) + a \quad J(g_{a}) = J(g) \]

\[ \text{critical for } f_{a} \neq 0 \quad g(x) = -a \]

\[ \text{non-sing for } f_{a} = 0 \quad J(g)(x) \neq 0 \quad \text{i.e. } -a \text{ is regular } \Rightarrow \text{SARD} \text{ They exist.} \]

On a manifold:

\[ \mathcal{M} \subset \mathbb{R}^{n} \]

not rel. value

\[ x_{1}, \ldots, x_{n} \text{ local ord. } f(x) (a_{n+1}, \ldots, a_{r}) \]

\[ f(x) = a_{1}x_{1} \cdot x_{2} \ldots a_{r}x_{r} \]

\[ a_{i} \neq 0, \quad f(a) = a_{i}x_{i} \quad \text{an } \text{ Morse on } x_{i} \]

\[ \Rightarrow a \cdot e \quad f(a) = a_{i}x_{i} + a_{r}x_{r} \text{ Morse} \]

\[ A_{i} = \{ a_{i}, f_{a}(x) \text{ non Morse on } x_{i}\} \]

\[ \text{A has non Morse } \Rightarrow (\text{SARD}) \]