TOPOLOGIA E GEOMETRIA DIFFERENZIALE

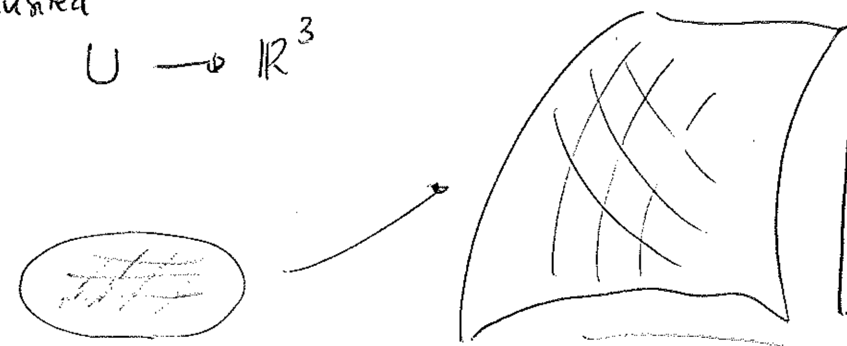
Prof. M. Spera a.a. 2009/10

Lezione VII

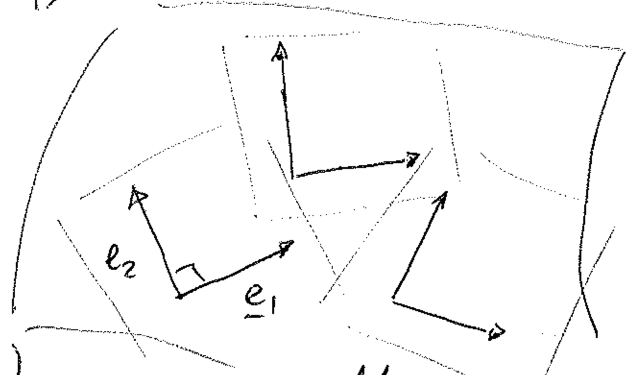
✱ La teoria delle superficie rivisitata

Surface theory revisited

$\underline{r} : U \to \mathbb{R}^3$



rif. di Darboux

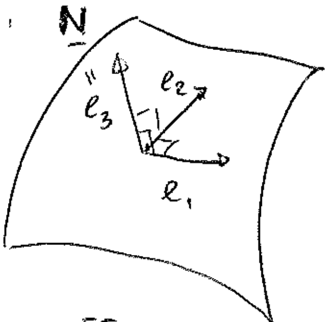


✱✱

Sia $(\underline{e}_1, \underline{e}_2, \underline{e}_3)$; $(\underline{e}_1, \underline{e}_2)$ base ortonormale di T_pM "diade"; $\underline{e}_3 = \underline{N}$

(collegabile tramite G-S)

"répère mobile" (Cartan)



$\bar{V} \equiv \frac{\partial \underline{r}}{\partial V} \ldots = v_1 \frac{\partial \underline{r}}{\partial u} + v_2 \frac{\partial \underline{r}}{\partial v}$

///

$\bar{V} = v_1 \underline{r}_u + v_2 \underline{r}_v = \tilde{v}_1 \underline{e}_1 + \tilde{v}_2 \underline{e}_2$

1-forme

$$(\diamond) \quad \boxed{d\underline{r} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2}$$

se $\bar{V} \in T_pM$, è $\bar{V} = \omega_1(\bar{V})\,\underline{e}_1 + \omega_2(\bar{V})\,\underline{e}_2$

$\langle d\underline{r}, \bar{V} \rangle = \frac{\partial \underline{r}}{\partial V}$

$\langle \bar{V}, \underline{e}_j \rangle = \omega_j(\bar{V})$

$\| $

$I(\bar{V}, e_j) \leftarrow$ 1ª f. fond. (metrica)

$\langle d\underline{r}, \bar{V} \rangle = (\omega_1 e_1 + \omega_2 e_2, \bar{V}) = \underbrace{\omega_1(\bar{V})}_{\tilde{v}_1} e_1 + \underbrace{\omega_2(\bar{V})}_{\tilde{v}_2} e_2$

$$(\diamond\diamond) \quad \boxed{d\underline{e}_i = \sum_{j=1}^{3} \omega_{ij}\, \underline{e}_j} \quad \text{1-forme}$$

(Differenziando $(\diamond)$ e $(\diamond\diamond)$ si ottengono le equazioni di struttura di Cartan ✱✱✱ v. oltre)

$$\boxed{d\underline{e}_i = \sum_j \omega_{ij}\, \underline{e}_j}$$

da $\langle e_i, e_j \rangle = \delta_{ij}$ si ha

$$0 = d\langle e_i, e_j\rangle = \langle de_i, e_j\rangle + \langle e_i, de_j\rangle$$

$$= \langle \sum_k \omega_{ik} e_k, e_j \rangle + \langle e_i, \sum_k \omega_{jk} e_k \rangle$$

$$= \omega_{ij} + \omega_{ji} \qquad \Rightarrow \quad \omega_{ij} = -\omega_{ij}$$

$$(\Rightarrow \omega_{ii} \equiv 0)$$

★ operatore di forma:

$$\underbrace{d\underline{e}_3}_{N} = \omega_{31}\, \underline{e}_1 + \omega_{32}\, e_2 = -\omega_{13}\, \underline{e}_1 - \omega_{23}\, e_2$$

$$\omega_{13}(v) = \mathrm{II}(v, e_1)$$

$$\omega_{23}(v) = \mathrm{II}(v, e_2)$$

$$\Rightarrow \boxed{\underbrace{-de_3}_{S} = \omega_{13}\, \underline{e}_1 + \omega_{23} e_2}$$

$$\boxed{\begin{cases} \omega_{13} = h_{11}\, \omega_1 + h_{12}\, \omega_2 \\ \omega_{23} = h_{21}\, \omega_1 + h_{22}\, \omega_2 \end{cases}}$$

$$S_p \sim \left(h_{ij}(p) \right)$$

da quanto già sappiamo, dobbiamo trovare $h_{ij} = h_{ji}$:

Tutta la teoria segue dalle equazioni di struttura (Cartan)

★ S_p è simmetrico $\Rightarrow$ la sua matrice rispetto ad una base ortonormale deve essere simmetrica

Equazioni di Struttura (E. Cartan - caso particolare)

$$\begin{cases} \text{(i)} \quad d\omega_1 = \omega_{12}\wedge\omega_2 \qquad d\omega_2 = -\omega_{12}\wedge\omega_1 \\[2ex] \text{(ii)} \quad d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik}\wedge\omega_{kj} \qquad i,j=1,2,3 \end{cases}$$

si ottengono da:

$$\underline{0} = d(d\underline{r}) = d\omega_1\underline{e}_1 + d\omega_2\underline{e}_2 - \omega_1\wedge\underbrace{\sum\omega_{1j}\underline{e}_j}_{d\underline{e}_1} - \omega_2\wedge\underbrace{\sum\omega_{2j}\underline{e}_j}_{d\underline{e}_2}$$

$$= (\underbrace{d\omega_1 - \omega_2\wedge\omega_{21}}_{=0})\,\underline{e}_1 + (\underbrace{d\omega_2 - \omega_1\wedge\omega_{12}}_{=0})\,\underline{e}_2 - (\underbrace{\omega_1\wedge\omega_{13} + \omega_2\wedge\omega_{23}}_{=0})\,\underline{e}_3$$

$\Rightarrow$ (i)

v. oltre

$$\underline{0} = d(d\underline{e}_i) = d\left(\sum_k \omega_{ik}\underline{e}_k\right) = \sum_k\left(d\omega_{ik}\underline{e}_k - \omega_{ik}\wedge\sum\omega_{kj}\underline{e}_j\right)$$

$$= \sum_{j=1}^{3} d\omega_{ij}\,\underline{e}_j - \sum_{j=1}^{3}\left(\sum_{k=1}^{3}\omega_{ik}\wedge\omega_{kj}\right)\underline{e}_j$$

$$= \sum_{j=1}^{3}\left[d\omega_{ij} - \sum_{k=1}^{3}\omega_{ik}\wedge\omega_{kj}\right]\underline{e}_j$$

$\Rightarrow$ (ii)

Intanto si ricava la simmetria dell'operatore di forma

infatti
$$\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23} = 0$$
$$\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2 \qquad \omega_{23} = h_{21}\omega_1 + h_{22}\omega_2$$

$$\Rightarrow \quad h_{12}\, \omega_1 \wedge \omega_2 + h_{21}\, \omega_2 \wedge \omega_1 = 0$$

$$\Rightarrow \quad (h_{12} - h_{21})\, \omega_1 \wedge \omega_2 = 0$$

$$\Rightarrow \quad \boxed{h_{12} = h_{21}}$$

Notiamo poi che se $\underline{e}_1$ ed $\underline{e}_2$ individuano direzioni principali

$$h_{12} = h_{21} = 0 \qquad e \qquad \omega_{13} = k_1\omega_1, \quad \omega_{23} = k_2\omega_2$$

principal directions (~ linee di curvatura)

Cosa succede "ruotando" $(\underline{e}_1, \underline{e}_2)$? $\quad \vartheta = \vartheta(p) \quad (*)$

$$\begin{cases} \bar{e}_1 = \cos\vartheta\, \underline{e}_1 + \sin\vartheta\, \underline{e}_2 \\ \bar{e}_2 = -\sin\vartheta\, \underline{e}_1 + \cos\vartheta\, \underline{e}_2 \end{cases} \qquad \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

$$d\underline{r} = \omega_1 \underline{e}_1 + \omega_2 \underline{e}_2 = \bar{\omega}_1 \bar{\underline{e}}_1 + \bar{\omega}_2 \bar{\underline{e}}_2$$

$$= \bar{\omega}_1 (\cos\vartheta\, e_1 + \sin\vartheta\, e_2) + \bar{\omega}_2(-\sin\vartheta\, \underline{e}_1 + \cos\vartheta\, \underline{e}_2)$$

$$= \overbrace{(\cos\vartheta\, \bar{\omega}_1 - \sin\vartheta\, \bar{\omega}_2)}^{\omega_1}\, \underline{e}_1 +$$

$$+ \underbrace{(\sin\vartheta\, \bar{\omega}_1 + \cos\vartheta\, \bar{\omega}_2)}_{\omega_2}\, \underline{e}_2$$

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} = \begin{pmatrix}\cos\vartheta & -\sin\vartheta\\ \sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}\bar\omega_1\\ \bar\omega_2\end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix}\bar\omega_1\\ \bar\omega_2\end{pmatrix} = \underbrace{\begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix}}_{=(*)}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

[come era da attendersi]

$$\begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix} \leftrightarrow e^{i\vartheta}$$

$$\left.\frac{d}{d\vartheta}\right|_{\vartheta=0}(\ ) = \left.\frac{d}{d\vartheta}\right|_{\vartheta=0}(\ ) = i$$

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$$

$$\sim \begin{pmatrix}0 & d\vartheta\\ -d\vartheta & 0\end{pmatrix} \leftrightarrow i\,d\vartheta$$

$\ast$ intuitivamente: $\bar\omega_{12} = \omega_{12} + d\vartheta$

$$d\bar e_1 = \bar\omega_{12}\,\bar e_2$$

$$\| $$

$$d(\cos\vartheta\, e_1 + \sin\vartheta\, e_2) = \bar\omega_{12}(-\sin\vartheta\, e_1 + \cos\vartheta\, e_2)$$

$$\|$$

$$\left[\begin{aligned}&-\sin\vartheta\, d\vartheta\, e_1 + \cos\vartheta\, de_1\\ &+\cos\vartheta\, d\vartheta\, e_2 + \sin\vartheta\, de_2\end{aligned}\right] = \left[\begin{aligned}&-\sin\vartheta\,\bar\omega_{12}\, e_1\\ &+\bar\omega_{12}\cos\vartheta\, e_2\end{aligned}\right]$$

$$\|$$

$$(-\sin\vartheta\, d\vartheta - \sin\vartheta\,\omega_{12})e_1 = -\sin\vartheta\,\bar\omega_{12}\, e_1$$

$$+(\qquad) \qquad + (\qquad)$$

$$\Rightarrow \quad \boxed{\bar\omega_{12} = \omega_{12} + d\vartheta}$$

$e_1 + i e_2 \equiv \xi$

$\bar e_1 + i\bar e_2 \equiv \bar\xi$

$\bar\xi = e^{i\vartheta}\xi$

$d\bar\xi = i e^{i\vartheta} d\vartheta\,\xi + e^{i\vartheta} d\xi$

$= i\,d\vartheta\cdot\bar\xi + e^{i\vartheta} d\xi$

$$\boxed{d\xi = i\,d\psi\,\xi}$$

$= i\,d\vartheta\cdot\bar\xi + e^{i\vartheta}\, i\,d\psi\,\xi$

$= i(d\vartheta + d\psi)\,\bar\xi$

variante, col formalismo complesso

Inoltre da $\begin{pmatrix}\bar{\omega}_1\\ \bar{\omega}_2\end{pmatrix} = \begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$

segue pure che $\begin{pmatrix}\omega_{13}\\ \omega_{23}\end{pmatrix} = \begin{pmatrix}\cos\vartheta & \sin\vartheta\\ -\sin\vartheta & \cos\vartheta\end{pmatrix}\begin{pmatrix}\omega_{13}\\ \omega_{23}\end{pmatrix}$ (chiaro...)

in particolare $\bar{\omega}_1 \wedge \bar{\omega}_2 = \omega_1 \wedge \omega_2$

$$\bar{\omega}_{13} \wedge \bar{\omega}_{23} = \omega_{13} \wedge \omega_{23}$$

Riprendiamo le prime equazioni di struttura

forma di connessione (di Levi-Civita)

$$\boxed{d\omega_{12} = -\omega_{13} \wedge \omega_{23}}$$ (Gauss) "cogauss"

$$d\omega_{13} = \omega_{12} \wedge \omega_{23}$$
$$d\omega_{23} = -\omega_{12} \wedge \omega_{13}$$
(Codazzi-Mainardi)

[Equazioni di compatibilità] "dazzi"

$$d\omega_{12} = -[h_{11}\omega_1 + h_{12}\omega_2] \wedge [h_{21}\omega_1 + h_{22}\omega_2]$$
$$= h_{21}h_{12}\,\omega_1 \wedge \omega_2 - h_{11}h_{22}\,\omega_1 \wedge \omega_2$$
$$= -[h_{11}h_{22} - h_{12}h_{21}]\,\omega_1 \wedge \omega_2$$
$$= -K\,\omega_1 \wedge \omega_2$$

$\det(h_{ij}) = K = k_1 k_2$ curvatura gaussiana

$$\boxed{d\omega_{12} = -K\omega_1 \wedge \omega_2}$$

ora $d\bar{\omega}_{12} = d(\omega_{12} + d\vartheta) = d\omega_{12} + \underbrace{d^2\vartheta}_{=0} = d\omega_{12}$

e $\omega_1 \wedge \omega_2 = \bar{\omega}_1 \wedge \bar{\omega}_2$

$(K = \bar{K})$

$\Rightarrow$ K non dip. dal riferimento e dato che ω_{12} dipende solo dalla metrica, ciò vale anche per K $\Rightarrow$ Theorema Egregium ★★★