Functional Analysis

Second part - a.y. 2012-13

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3-dec-2012 (2 hrs). The space $\mathcal{L}(E,F)$ of bounded linear operators between two Banach spaces E,F. Operator norm $||T||_{\mathcal{L}} = \sup\{||Tv||_F,\ ||v||_E \le 1\}$. Various notions of convergence for a sequence of operators $T_n \in \mathcal{L}(E,F)$ to $T \in \mathcal{L}(E,F)$: uniform ($||T_n - T||_{\mathcal{L}} \to 0$), strong ($T_n v \to Tv$ in F $\forall\ v \in F$), weak ($< \phi, T_n v > \to < \phi, Tv >$ $\forall v \in F$, $\forall \phi \in F'$). The Weierstrass criterion for uniform convergence of series of operators. Neumann series: for $T \in \mathcal{L}(E)$ and $||T||_{\mathcal{L}} < 1$, $(I-T)$ is invertible (and hence $(I-T)^{-1} \in \mathcal{L}(E)$). Moreover, $(I-T)^{-1} = \sum_{n=0}^{+\infty} T^n$. It follows that the subset of invertible operators is *open* in $\mathcal{L}(E)$: if T is invertible then for any $S \in \mathcal{L}(E)$ such that $||S|| < ||T^{-1}||$, then $T+S$ is invertible. Adjoint operator $T^* \in \mathcal{L}(F', E')$. It is defined by the identity $< T^*\phi, v > = < \phi, Tv >$ for any $v \in F$, $\phi \in F'$. It holds $\|T^*\| = \|T\|$, as a consequence of Hahn-Banach. In case $E = F = H$ a Hilbert space, from the identification $H \equiv H'$ given by the Riesz representation theorem, one considers $T, T^* \in \mathcal{L}(H)$. If $T = T^*$ the operator is called *self-adjoint* or *symmetric*.

Elements of spectral theory for $T \in \mathcal{L}(E)$. Resolvent set $\rho(T) \subset \mathbb{C}$: we have $\lambda \in \rho(T)$ if $(\lambda I - T)^{-1} \in \mathcal{L}(E)$. The resolvent set is open in $\mathbb{C}$. Moreover, if $|\lambda| > \|T\|$ then $\{\lambda \in \mathbb{C},\ |\lambda| > ||T||\} \subset \rho(T)$. Actually, denoting $r(T) = \limsup_n (||T^n||)^{1/n} \le ||T||$ the spectral radius of T, we have $\{\lambda \in \mathbb{C},\ |\lambda| > r\} \subset \rho(T)$.

5-dec-2012 (2hrs). Spectrum $\sigma(T) = \mathbb{C} \setminus \rho(T)$ of $T \in \mathcal{L}(E)$: it is a closed set contained in $B(0, ||T||) \subset \mathbb{C}$. Let $\lambda \in \sigma(T)$: If $\ker(T - \lambda I) \ne 0$ then λ is an eigenvalue of T, and belongs to the point spectrum. Otherwise, λ belongs to the continuous spectrum ($\ker(T - \lambda I) = 0$ but $(T - \lambda I)$ is not surjective). In particular, the map $(T - \lambda I)^{-1}$ may be defined either in a dense or in a proper closed subspace of E. Examples: the right shift τ_r in ℓ^1 (or ℓ^2), or the diagonal operator $T_\alpha : \{x_n\} \mapsto \{\alpha_n x_n\}$ where $0 \ne \alpha_n \to 0$. In both cases 0 belongs to the continous spectrum. Moreover, $\{\alpha_n\} \subset \sigma(T_\alpha)$ is the point spectrum of T_α, while the point spectrum of τ_r is empty.

Resolvent operator $R_\lambda = (T - \lambda I)^{-1}$ of $T \in \mathcal{L}(E)$, with $\lambda \in \rho(T)$. Resolvent equation $R_\lambda - R_\mu = (\lambda - \mu) R_\lambda R_\mu$: it yields $\frac{dR_\lambda}{d\lambda} = R_\lambda^2$, that is $\lambda \mapsto R_\lambda$ is a holomorphic function, whose singularities are in $\sigma(T)$. In particular, the Cauchy integral formula (and the calculus of residues) involving R_λ and a given holomorphic function $f(z)$

allows to consistently define $f(T)$ (in particular, if $f(z) = e^z$, we obtain a formula for $\exp(T)$, while if $f(z) = 1$ we derive some information on the Jordan blocks of T).

The space $\mathcal{K}(E, F) \subset \mathcal{L}(E, F)$ of compact operators. Uniform limits of compact operators in $\mathcal{L}(E, F)$ are compact, i.e. $\mathcal{K}(E, F)$ is closed in $\mathcal{L}(E, F)$. A compact operator (right- or left-) composed with a bounded operator is compact. In particular, $\mathcal{K}(E) \equiv \mathcal{K}(E, E)$ is a bilateral ideal of $\mathcal{L}(E)$. The identity map is compact if and only if E is finite dimensional. Operators whose range is finite dimensional are compact: they are called *finite rank* operators. Limits of sequences of finite rank operators are compact.

Finite rank approximation for $T \in \mathcal{K}(E, H)$, with H a Hilbert space: given $v_1, ..., v_N \in H$ a ϵ-net for $T(B_E)$, set V_N =span$\langle v_1, ..., v_N\rangle$ and $T_N = P_N \cdot T$, where P_N is the orthogonal projection on V_N. We have that T_N has finite rank and $||T_N - T||_{\mathcal{L}(E,H)} \leq 2\epsilon$.

Examples: $T : (a_n)_n \mapsto (2^{-n} a_n)_n$ is compact on ℓ^1 as uniform limit of the finite rank operators $T_N : (a_n)_n \mapsto (\sigma_n 2^{-n} a_n)_n$, where $\sigma_n = 1$ for $n \leq N$ and $\sigma_n = 0$ for $n > N$.

10-dec-2012 (1hr). Some properties of compact operators: any injective $T \in \mathcal{K}(E)$ doesn't admit a bounded inverse. If E is reflexive (e.g. a Hilbert space), T is compact if and only if for any $v_n \rightharpoonup v$ weakly in E it holds $Tv_n \to Tv$ strongly in E. In particular, $T(\bar{B_E}) = \bar{T(B_E)}$. If $T \in \mathcal{K}(H)$ then $T^* \in \mathcal{K}(H)$ and conversely. Some examples of compact operators: integral (kernel-based) operator of Fredholm-Volterra type on $C^0([a, b])$. Operators based on Mercer kernels on a compact metric measure space: they are used for statistical learning purposes. Hilbert-Schmidt operators: if $K \in L^2([a, b] \times [a, b])$ then if $(Tx)(s) = \int_a^b K(s, t)x(t)\, dt$ we have $||T||_{\mathcal{L}} \leq ||K||_{L^2}$. Given a Hilbert basis (i.e. a complete orthonormal system) $\{\phi_n\}$ of $L^2([a, b])$, set $\psi_{nm}(s, t) = \phi_n(s)\phi_m(t)$: the elements ψ_{nm} are a Hilbert basis of $L^2([a, b] \times [a, b])$. Expand $K(s, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} k_{nm}\psi_{nm}(s, t)$, and setting respectively $K_N(s, t) = \sum_{n=1}^{M} \sum_{m=1}^{N} k_{nm}\psi_{nm}(s, t)$ and $(T_N x)(s) = \int_a^b K_N(s, t)x(t)\, dt$, we have $||T_N - T||_{\mathcal{L}} \leq ||K_N - K||_2 \to 0$, hence $T \in \mathcal{K}(L^2([a, b]))$ as limit of finite rank operators.

11-dec-2012 (2 hrs). The compact embedding $i : W^{1,p}([a, b]) \to C^0([a, b])$, $p > 1$; given a family of maps $u \in W^{1,p}([a, b])$ with equibounded norm, i.e. $\|u\|_p + \|u'\|_p \leq M$, we show that this family is equibounded in $C^0([a, b])$ and (uniformly) equicontinuous, hence by Ascoli-Arzela it is relatively compact in $C^0([a, b])$: decompose $u = v + c$, where $c = \frac{1}{b-a} \int_a^b u(x)\, dx$. We have, by Hölder inequality,

$$|c| \leq \frac{1}{|b-a|^{1/p}} \|u\|_p \leq \frac{1}{|b-a|^{1/p}} M, \quad |v(x)| \leq \int_{x_0}^{x} |u'(t)|\, dt \leq |b-a|^{1/p} \|u'\|_p \leq |b-a|^{1/p} M\,,$$

where $x_0 \in [a, b]$ is such that $v(x_0) = \frac{1}{b-a} \int_a^b v(x)\, dx = 0$. We deduce the uniform bound $\|u\|_\infty \leq CM$, where C depends only on $[a, b]$. To prove equicontinuity, observe that

$$|u(x) - u(y)| \leq \int_x^y |u'(t)|\, dt \leq |x - y|^\alpha \|u'\|_p \leq |x - y|^\alpha M, \qquad \text{where } \alpha = 1 - \frac{1}{p}\,,$$

hence the maps u are equi-Hölder continuous.

The Fredholm Alternative for operators of the type $A = I - T$, with $T \in \mathcal{K}(H)$, H a Hilbert space: $\ker A$ is finite dimensional, the range $R(A)$ is closed, hence there holds the orthogonal direct sum decomposition $H = R(A) \oplus \ker A^* = R(A^*) \oplus \ker A$, where $A^* = I - T^*$. Moreover, $\ker A = 0 \Leftrightarrow R(A) = H$, and finally $\dim \ker A$ =$\dim \ker A^* < +\infty$.

The Fredholm alternative gives a procedure to solve equation $Au = u - Tu = f$. First of all, solve the associated adjoint homogeneous equation, i.e. the fixed point equation $v = T^* v$. If the solution is trivial, then $Au = f$ admits a unique solution for any datum $f \in H$. Otherwise, call $v_1, ...v_k$ a basis of $\ker A^*$ (i.e. a maximal independent set of fixed points of T^*); then there are solutions of $Au = f$ provided f verifies the orthogonality conditions $< f, v_i >= 0$ for any $i = 1, ..., k$.

The Fredholm Alternative holds more generally for operators of the type $A = I - T$ with $T \in \mathcal{K}(E)$, E a Banach space.

12-dec-2012 (2hrs). Spectrum of a compact operator: $0 \in \sigma(T)$ and $\sigma(T) \setminus \{0\}$, if non empty, is made of at most countably many eigenvalues λ_n, with $\lambda_n \to 0$ as $n \to +\infty$. The corresponding eigenspaces $\ker(\lambda_n I - T) \neq 0$ are finite-dimensional. If the operator is self-adjoint on a Hilbert space, then the eigenvalues are real, and $\max |\lambda_n| = \|T\|_{\mathcal{L}}$.

Spectral theory for self-adjoint compact operators in Hilbert spaces: the eigenvalues are real and there exists a Hilbert basis made of eigenvectors, which "diagonalizes" the operator. In particular, for $T \in \mathcal{K}(H)$, $T^* = T$, and e_n a orthonormal basis of eigenvectors, i.e. $Te_n = \lambda_n e_n$ (with $\lambda_n \to 0$), we have the diagonal representation $Tv = T(\sum_n c_n e_n) = \sum_n \lambda_n c_n e_n$, i.e. the operator can be identified with $\tilde{T} \in \mathcal{K}(\ell^2)$ given by $\tilde{T}(c_n) = (\lambda_n c_n)$.

Proof of the spectral theorem: we consider a (iterated) contrained optimization problem on the unit closed ball $\overline{B} = \{\|v\| \leq 1\}$ of H for the quadratic form $Q(v) = \langle Tv, v \rangle$ associated to $T \in \mathcal{K}(H)$. Notice first that $Q(v)$ is weakly continuous, since $v_n \rightharpoonup v_0$ implies $Tv_n \to Tv_0$, and moreover $\|v_n\| \leq M$ (weakly convergent sequences are bounded), whence

$$| \langle Tv_n, v_n \rangle - \langle Tv_0, v_0 \rangle | \leq |Tv_n - Tv_0| \cdot |v_n| + | \langle Tv_0, v_n - v_0 \rangle | \to 0.$$

By Weierstrass Theorem, $|Q(v)|$ reaches its maximum on the unit closed ball $\overline{B}$, which is weakly compact. Let e_1 be a maximum point. We have necessarily $\|e_1\| = 1$ because $Q(\lambda v) = \lambda^2 Q(v)$ for $\lambda \in \mathbb{R}$. Moreover, for any $e \in H$ such that $\|e\| = 1$ and $\langle e, e_1 \rangle = 0$, one has $\langle e, Te_1 \rangle = 0$, since by the Lagrange multipliers theorem e_1 is a critical point of the function $Q(v) + \lambda \|v\|^2 = \psi(\alpha, \beta, \lambda)$, where $v = \alpha e_1 + \beta e_2$ belongs to the 2-dimensional space spanned by e_1 and e. In particular, one deduces $Te_1 = \langle Te_1, e_1 \rangle \cdot e_1 = \lambda_1 e_1$, i.e. e_1 is an eigenvector of T and $|Q(e_1)| = |\langle Te_1, e_1 \rangle| = |\lambda_1|$, i.e. the eigenvalue λ_1 has maximum modulus among the eigenvalues of T (actually we have $|\lambda_1| = \|T\|_{\mathcal{L}}$).

Iterating this procedure, one obtains, for $n \geq 1$, an eigenvector e_n of T, with $\|e_n\| = 1$, and such that $< e_n, e_m >= 0$ for any $m < n$, corresponding to the maximum point of $|Q(v)|$ on $(\text{span}\{e_1, ..., e_{n-1}\})^\perp \cap \overline{B}$, with $\lambda_n = Q(e_n)$ the corresponding eigenvalue. Moreover, it holds $|\lambda_{n-1}| \geq |\lambda_n|$.

If for some $n_0 \in \mathbb{N}$ one has $\lambda_{n_0} = Q(e_{n_0}) = 0$, then $(\text{span}\{e_1, ..., e_{n_0-1}\})^\perp = \ker T$. Indeed, $Q(w) = 0$ for any $w \in (\text{span}\{e_1, ..., e_{n_0-1}\})^\perp$, and if $\langle w, e_i\rangle = 0 \; \forall \, i < n_0$, then $\langle Tw, e_i\rangle = \langle v, Te_i\rangle = 0$, i.e. also $Tw \in (\text{span}\{e_1, ..., e_{n_0-1}\})^\perp$. The polarization identity $4\langle Tv, u\rangle = Q(u + v) - Q(u - v)$ hence implies that $4\langle Tw, Tw\rangle = Q(w + Tw) - Q(w - Tw) = 0$ for any $w \in (\text{span}\{e_1, ..., e_{n_0-1}\})^\perp$, i.e. $Tw = 0$.

We deduce in this case that the set $\{e_1, ..., e_{n_0}\}$, completed with a (complete) orthonormal system of $\ker T$ yields a Hilbert basis of eigenvectors of T.

Otherwise, we are left with a orthonormal sequence $\{e_n\}_n$, so that in particular $e_n \rightharpoonup 0$ by Bessel inequality (for any $w \in H$, $\sum_n \langle e_n, w\rangle^2 \leq ||w||^2 \Rightarrow \langle e_n, w\rangle \to 0$ as $n \to +\infty$), and hence $|\lambda_n| = |Q(e_n)| \searrow 0$ by weak continuity of Q. Let $N = \overline{\text{span}\{e_1, ..., e_n, ...\}}^\perp$. For any $w \in N$ one necessarily has $|Q(w)| \leq |Q(e_n)|$ for any $n \in \mathbb{N}$, hence $Q(w) = 0$ and $N = \ker T$.

In this case, the set $\{e_n\}_{n\in\mathbb{N}}$, completed with a (complete) orthonormal system of $\ker T$ yields a Hilbert basis of eigenvectors of T. □

13-dec-12 (2hrs). Application of the spectral theorem to the representation of the (unique) solution of the homogeneous Dirichlet problem $\Delta u = f$ in $\Omega \subset \mathbb{R}^n$, $u = 0$ on $\partial\Omega$, where $f \in C^0(\Omega)$.

Denoting by $T : C^0(\Omega) \to C^2(\Omega)$ the solution operator $f \mapsto u$ (T corresponds, roughly speaking, to Δ^{-1}) one may extend it to an operator $T \in \mathcal{L}(L^2(\Omega))$ which happens to be compact and self-adjoint.

If $\{e_k(x)\}_k$ is a Hilbert basis of $L^2(\Omega)$ of eigenvectors of T with eigenvalues $\mu_k \in \mathbb{R}$ with $\mu_k \to 0$, then $\Delta e_k = \lambda_k e_k$ with $e_k(x) = 0$ on $\partial\Omega$ and $\lambda_k = 1/\mu_k$ (in particular $|\lambda_n| \to +\infty$). If Ω has a particularly simple shape (e.g. a circle or a rectangle in $\mathbb{R}^2$, a ball or a parallelopiped in $\mathbb{R}^n$), one may find $e_k(x)$ and the corresponding λ_k using the technique of separation of variables, that consists in seeking a solution $e(\xi_1, ..., \xi_n)$ of $\Delta_\xi e = \lambda e$ of the form $e(\xi_1, ..., \xi_n) = \Pi_{i=1}^n \Xi_i(\xi_i)$, where $\xi_1, ..., \xi_n$ is a "natural" coordinate system for the domain Ω. The partial differential equation solved by e translates in a ordinary differential equation for $\Xi_1, ..., \Xi_n$ and determines at once $e_k(x)$ and λ_k.

Having constructed the Hilbert basis $\{e_k(x)\}$ of $L^2(\Omega)$ which diagonalizes T, and expanding $f(x) = \sum_{k=1}^\infty f_k e_k(x)$, we obtain the following representation for the solution of $\Delta u = f$ in Ω, $u = 0$ on $\partial\Omega$:

$$u(x) = Tf(x) = T\left(\sum_{k=1}^\infty f_k e_k(x)\right) = \sum_{k=1}^\infty \frac{f_k}{\lambda_k} e_k(x)\,.$$

The Lax-Milgram Lemma: given a bilinear form $a(u, v)$, continuous ($a(u, v) \leq M\|u\|\|v\|$) and coercive ($0 < \alpha\|u\|^2 \leq a(u, u) \; \forall \; u \neq 0$) on a Hilbert space H, for any bounded linear form $\phi \in H^*$ there exists a unique $u \in H$ such that $a(u, v) = \phi(v)$ for any $v \in H$. In particular, $\|u\| \leq \alpha^{-1}\|\phi\|_*$.

If moreover a is symmetric (i.e. $a(u, v) = a(v, u)$), we have the characterization $u = \arg\min\{\frac{1}{2}a(v, v) - \phi(v)\,,\ v \in H\}$.

Proof: by Riesz representation theorem, the equation to be solved can be rewritten as $\langle Au, v\rangle = \langle f, v\rangle$ for any $v \in H$, i.e. $Au = f$, where $A \in \mathcal{L}(\mathcal{H})$ verifies the estimates $0 < \alpha\|u\| \leq \|Au\| \leq M\|u\|\ \forall\ u \neq 0$.

From $\alpha\|u\| \leq \|Au\|$ (which is called an *a priori estimate*) it follows that $\ker A = 0$. Moreover, $\alpha\|u_n - u_m\| \leq \|Au_n - Au_m\|$ implies that if $y_n = Au_n \to y$ in H, i.e. Au_n is a Cauchy sequence in H, then also u_n is a Cauchy sequence, hence $u_n \to u$ in H by completeness, thus yielding $y = Au$. One concludes that A has a closed range $R(A)$ in H. Finally, if $v \perp R(A)$, then $\langle v, Au\rangle = 0\ \forall\ u \in H$. In particular, choosing $u = v$, we have $0 = \langle v, Av\rangle \geq \alpha\|v\|^2$, thus $v = 0$ and $R(A) = H$. We just proved that A is both injective and surjective, and the conclusion of the Lemma follows.

In case of a symmetric a, since $\alpha\|u\|^2 \leq a(u, u) \leq M\|u\|^2$, the scalar product $((u, v)) := a(u, v)$ is equivalent to $\langle\cdot,\cdot\rangle$, hence by Riesz representation theorem applied to H endowed with $((\cdot,\cdot))$, one has $\phi(v) = a(g, v)$ for a certain $g \in H$, whence u verifies $a(u - g, v) = 0\ \forall\ v \in H$, i.e. u is the orthogonal projection (with respect to the scalar product induced by a) of g on H, in other words u minimizes the (squared) distance (induced by a) $a(v-g, v-g)$, or, equivalently, the quadratic functional $F(v) = \frac{1}{2}a(v, v) - \phi(v)$, for $v \in H$, whose Euler-Lagrange equation $\partial_v F(u) \equiv \langle F'(u), v\rangle = 0$ for any direction $v \in H$ is precisely given by $a(u, v) = \phi(v)$ for any $v \in H$.

□

The Galerkin approximation method: if $V_h \subset H$, $\dim V_h < +\infty$, one considers the solution u_h of the system $a(u, v) = \phi(v)\ \forall\ v \in V_h$. The Lemma of Céa guarantees that $\|u - u_h\| \leq \frac{M}{\alpha}\mathrm{dist}\,(u, V_h)$ (in other words, u_h is comparable to the orthogonal projection of u on V_h): indeed, $a(u - u_h, u - u_h) = a(u - u_h, u - v)$ for any $v \in V_h$ since $a(u, v - u_h) = a(u_h, v - u_h) = \phi(v - u_h)$, whence $\alpha\|u - u_h\|^2 \leq M\|u - u_h\|\|u - v\|$ for any $v \in V_h$ and the conclusion follows.

Hence, considering a sequence of finite-dimensional spaces $V_h \subset V_{h+1}$ such that $H = \overline{\cup_h V_h}$, one has the convergence $u_h \to u$ in H as $h \to +\infty$.

Remark that the approximating finite-dimensional problem is a linear system with a positive definite coefficients matrix, called *stiffness matrix*, which is given by $[a(f_i, f_j)]$, with $\{f_i\}$ a basis for V_h.

The choice of the sequence V_h invading H and of a basis $\{f_i\}$ for V_h is aimed to efficiently solve the approximating linear system, and also to have the best possible convergence rate for the error estimate $||u_h - u||$. Here are some examples in case $H = L^2(\Omega)$, $\Omega \subset \mathbb{R}^n$:

1) if a is represented by a compact self-adjoint operator, then considering a Hilbert basis $\{e_n\}_{n\in\mathbb{N}}$ of $L^2(\Omega)$ made of eigenvectors, and setting $V_h =$span$< e_1, ..., e_h >$, the corresponding system is diagonal.

2) considering a basis $\{f_i\}$ of V_h made of *finite elements* (piecewise linear or polyno-

mial function insisting on a fixed triangulation of the domain) yields a sparse stiffness matrix. Finite elements are used in numerical fluid dynamics, material science, elasticity,...

3) Haar basis, wavelets, radial basis functions: these Hilbert basis of $L^2(\Omega)$ are used in signal and image processing and statistical analysis, being not computationally expensive, and also since they are able to take into account localized oscillation phenomena at any scale in physical and in frequency space.

3) if the original problem admits a smooth solution (for example, $u \in C^\infty(\Omega)$ as for Laplace equation), it may be convenient to use *spectral methods* for its approximation, i.e. to consider a Hilbert basis of $L^2(\Omega)$ made of orthogonal polynomials (e.g. the trigonometric system, the Legendre polynomials, the Hermite polynomials): since the Lemma of Céa states that the error estimate $\|u - u_h\|$ is comparable to the distance of u to its orthogonal projection on V_h, hence the convergence rate will be better according to the regularity of u (for instance, the more regular u, the more rapidly its Fourier coefficients decay to 0).

16-dec-2012 (2 hrs). Integration by parts formula in $\mathbb{R}^n$ and notion of weak (partial) derivatives. Example: the function $|x|$, for $x \in \mathbb{R}$, has the Heaviside function as weak derivative. Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $W^{1,p}(I)$, $I \subset \mathbb{R}$ an interval, and $W^{1,p}(\Omega)$. Definition of $W^{1,p}(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ w.r.t. the $W^{1,p}$ norm $\|u\|_{W^{1,p}} = \|u\|_{L^p} + \|\nabla u\|_{L^p}$. Definition of $W^{1,p}(\Omega)$ as the space of those $u \in L^p(\Omega)$ admitting weak derivatives in $L^p(\Omega)$. Continuity properties of Sobolev functions in $I \subset \mathbb{R}$. Some properties of $W^{1,p}(\Omega)$: completeness, reflexivity, separability, according to the exponent p.

18-dec-2012 (2 hrs). Density of smooth functions in $W^{1,p}(\Omega)$: extension of a function $u \in W^{1,p}(\Omega)$ to a function $\bar{u} \in W^{1,p}(\mathbb{R}^n)$ and regularization by convolution. Leibniz' rule and chain rule for Sobolev maps. The Hilbert space $H^1 = W^{1,2}$. Characterization of maps in $W^{1,p}(\Omega)$: uniformly bounded differential quotients w.r.t the L^p norm, weak derivative as a bounded linear functional on $L^{p'}$: in case $p = 1$ these properties characterize the space $BV(\Omega)$ of functions of bounded variation, i.e. the functions whose weak partial derivatives are represented by Radon measures. Example: the Heaviside function has the Dirac mass δ_0 as weak derivative. The space $W_0^{1,p}(\Omega)$, Poincaré inequality.

19-dec-2012 (2 hrs). First written test.

14-jan-2013 (2 hrs). Weak / variational formulation of elliptic boundary value problems in dimension 1. Classical vs weak solutions. Homogeneous Dirichlet problem: weak formulation in H_0^1, existence, uniqueness, a priori estimates, H^2-regularity and higher regularity of the weak solution of the Variational characterization of the weak solution as the minimizer of the Dirichlet energy. Analysis of the (homogeneous) Sturm-Liouville problem, spectral decomposition of Sturm-Liouville operators, spectral approximation methods. Homogeneous Neumann problem: weak formulation in

H^1, existence, uniqueness, a priori estimates, variational characterization of the weak solution, higher regularity.

16-jan-2013 (2 hrs). Non homogeneous Dirichlet and Neumann problems, Stampacchia theorem. Examples with mixed / periodic boundary conditions, and with non symmetric associated bilinear form. Maximum principle for the Dirichlet problem, Stampacchia's truncation method. Corollaries: comparison principles, stability in L^∞ for the solution with respect to the data.

21-jan-2013 (2 hrs). Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $W^{1,p}(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open bounded of class C^1. Critical exponent p^*. Sobolev embedding theorems in the subcritical, critical (Sobolev-Gagliardo-Nirenberg) and supercritical (Sobolev-Morrey) case. Sobolev embedding in the case $p = 1$ and isoperimetric inequality in $\mathbb{R}^n$.

Rellich-Kondrachov compact embedding for $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^n$ open bounded of class C^1. The space $W^{1,p}_0(\Omega)$ with $\Omega \subset \mathbb{R}^n$ open bounded, Poincaré inequality.

Weak and variational formulation of elliptic boundary value problems in $\Omega \subset \mathbb{R}^n$: formulation in H^1_0 of the homogeneous Dirichlet problem with datum $f \in L^2$, existence, uniqueness, a priori estimates for the weak solution via Lax-Milgram. Variational characterization as the minimizer of the energy $E(v) = \|\nabla v\|_2^2 + \|v - f\|_2^2$ on $H^1_0(\Omega)$. Elliptic problems in divergence form.

22-jan-2013 (2 hrs). Solvability of a general second-order elliptic boundary value problem in divergence form through the Fredholm Alternative. Nonhomogeneous Dirichlet problem. Homogeneous Neumann problem. Maximum principle for elliptic equations. Spectral theory of the Laplacian: existence of a Hilbert basis of $L^2(\Omega)$ made by eigenfunctions of the Laplacian in $H^1_0(\Omega)$. The heat equation and the wave equation: methods of resolution through discretization in time (Euler-type schemes) or through finite-dimensional approximation of the Laplacian (Galerkin method, corresponding to the technique of separation of variables). Gradient flow aspects of the heat equation. Discretized gradient flow of the Dirichlet energy as an example of Tychonoff regularization in machine learning theory.

23-jan-2013 (2hrs). Identification of $\mathcal{M}(\Omega)$, the space of Radon measures on $\Omega \subset \mathbb{R}^n$ with the dual space $C^0_c(\Omega)'$ via Riesz representation theorem. Weak-* convergence on a dual space and Banach-Alaloglu compactness theorem. Application to the existence of solutions to minimum norm problems in dual spaces via direct methods (i.e. weak-* compactness of minimizing sequences plus lower semicontinuity of the norm).

Definition and characterization of the space $BV(\Omega)$ (function of bounded variation): $u \in BV(\Omega)$ if $u \in L^1(\Omega)$ and the (distributional) gradient $Du = (D_1 u, ..., D_n u)$ is a (vector) Radon measure, which satisfies the integration by part formula (Gauss-Green)

$$\int_\Omega u \,\mathrm{div}\,\vec{\phi} = -\int_\Omega \vec{\phi} \cdot dDu \qquad \text{for any } \vec{\phi} \in [C^0_c(\Omega)]^n \,.$$

Total variation of a vector Radon measure: for $\vec{\mu} = (\mu_1, ..., \mu_n)$ with $\mu_i \in \mathcal{M}(\Omega) = (C_c^0(\Omega))'$ we have the decomposition $\vec{\mu} = \vec{\nu}|\vec{\mu}|$, where $|\vec{\mu}|$ is a positive measure (called the total variation measure) and $|\vec{\nu}(x)| = 1$ for $|\vec{\mu}|$ a.e. $x \in \Omega$. The total variation of $\vec{\mu}$ is defined as

$$\|\vec{\mu}\| = \sup \left\{ \int_\Omega \vec{\phi} \cdot d\vec{\mu} = \int_\Omega \vec{\phi} \cdot \vec{\nu}\, d|\vec{\mu}|\,, \quad \vec{\phi} \in [C_c^0(\Omega)]^n, \quad \|\vec{\phi}\|_\infty \leq 1 \right\} = |\vec{\mu}|(\Omega).$$

Example: the characteristic function $\mathbf{1}_E$ of an open bounded set $E \subset \mathbb{R}^n$ with $\partial E \cap \Omega$ of class C^1 belongs to $BV(\Omega)$, since by Gauss-Green formula

$$D\mathbf{1}_E(\vec{\phi}) = -\int_E \operatorname{div} \vec{\phi}\, dx = -\int_{\partial E} \vec{\phi} \cdot \vec{n}\, d\sigma\,,$$

where $\vec{n}$ is the unit outer normal to ∂E and $d\sigma$ is the surface measure on ∂E, so that $|D\mathbf{1}_E(\vec{\phi})| \leq \|\vec{\phi}\|_\infty \cdot \text{Area}(\partial E \cap \Omega)$, i.e. $D\mathbf{1}_E$ is a vector Radon measure, and in particular $D\mathbf{1}_E = \vec{\nu}|D\mathbf{1}_E|$, where $\vec{\nu}(x) = -\vec{n}(x)$ is the inner unit normal to $\partial E \cap \Omega$ and $|D\mathbf{1}_E| = d\sigma$. By a suitable choice of the test function $\vec{\phi}$ in such a way that $|\vec{\phi}(x)| \leq 1$ and $\vec{\phi} = -\vec{n}$ on $\partial E \cap \Omega$ one gets $|D\mathbf{1}_E|(\Omega) = \text{Area}(\partial E \cap \Omega)$.

Definition of finite perimeter (or Caccioppoli) sets in Ω: they are Lebesgue measurable sets $E \subset \Omega$ such that $P_\Omega(E) \equiv |D\mathbf{1}_E|(\Omega) < +\infty$, i.e. $\mathbf{1}_E \in BV(\Omega)$. Isoperimetric and isovolumetric problem in $\mathbb{R}^n$. Weak formulation of the isovolumetric problem in the class of finite perimeter sets: fix $R > 1$ (sufficiently large) and set

$$\mathcal{P} = \left\{ E \subset B_R(0),\, \mathcal{L}^n(E) = \int_{\mathbb{R}^n} \mathbf{1}_E\, d\mathcal{L}^n = 1,\; \mathbf{1}_E \in BV(B_{2R}(0)) \right\},$$

i.e. $\mathcal{P}$ contains sets $E \subset B_R(0)$ having unit volume and finite perimeter $\|D\mathbf{1}_E\| \equiv |D\mathbf{1}_E|(B_{2R}(0))$ in $B_{2R}(0)$: observe that since $E \subset B_R(0)$, the perimeter of E in $B_{2R}(0)$ coincides with the whole perimeter of E in $\mathbb{R}^n$, i.e. with $|D\mathbf{1}_E|(\mathbb{R}^n)$. Consider the isovolumetric problem

$$\min_{E \in \mathcal{P}} \|D\mathbf{1}_E\|\,.$$

If $E_n \in \mathcal{P}$ is a minimizing sequence, i.e. $\|D\mathbf{1}_{E_n}\| \to \inf_{F \in \mathcal{P}} \|D\mathbf{1}_F\|$, we have

$$\|\mathbf{1}_{E_n}\|_{BV(B_{2R}(0))} = 1 + \|D\mathbf{1}_{E_n}\| \leq C,$$

so that, up to a subsequence, $\mathbf{1}_{E_n} \to \mathbf{1}_E$ in $L^1(B_{2R}(0))$ by the compact embedding of $BV(B_{2R}(0))$ in $L^1(B_{2R}(0))$ (Rellich Theorem). We deduce $E \subset B_R(0)$ and $\mathcal{L}^n(E) = 1$. Moreover, passing to the limit in the integration by part formula, we have $D\mathbf{1}_{E_n}(\vec{\phi}) \to D\mathbf{1}_E(\vec{\phi})$ for any $\vec{\phi} \in [C_c^0(\mathbb{R}^n)]^n$ and

$$\|\nabla\mathbf{1}_E\| \leq \liminf_{n \to +\infty} \|\nabla\mathbf{1}_{E_n}\| = \inf_{F \in \mathcal{P}} \|\nabla\mathbf{1}_F\|$$

by lower semicontinuity of the total variation norm. Hence E has minimum perimeter in the class $\mathcal{P}$.

The regularity theory (based for example on Steiner symmetrization) allows to conclude that the optimal set E is the unit volume round ball in $\mathbb{R}^n$.

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