— Lectures on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture VII

As usual $n \geq 1$

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★ Differential k-forms

Let $\Lambda^k_p(\mathbb{R}^n)$ denote the — $\binom{n}{k}$ dimensional — space of k-forms on $T_p\mathbb{R}^n$ [notice the slight change of notation]

A basis thereof is given by $\left(dx^{i_1}\big|_p \wedge \dots dx^{i_k}\big|_p \right)$

$$i_1 < i_2 < \dots < i_k \quad , \quad i_j \in \{1, 2, \dots, n\}$$

notice the upper indices appended to coordinates: this is in view of applying tensor notation

Obviously

$$dx^{i_1}\big|_p \wedge \dots dx^{i_k}\big|_p = dx^{i_1} \wedge \dots dx^{i_k}\big|_p$$

Also set $dx^{i_1} \wedge \dots dx^{i_k} =: dx^I$ $\quad I = (i_1 \dots i_k)$

(abridged notation)

The differential k-forms (on $\mathbb{R}^n$) are then the (smooth) functions

$$\omega: \quad \mathbb{R}^n \ni p \longmapsto \omega_I(p)\, dx^I$$

Einstein

$$= \omega_{i_1 \dots i_k}(p)\, dx^{i_1} \wedge \dots dx^{i_k}\big|_p$$

sometimes, for clarity, Σ will be added

$\omega_I \equiv \omega_{i_1 \dots i_k} \in \Lambda^0(\mathbb{R}^n)$
smooth function

Notation: $\Lambda^k(\mathbb{R}^n)$

Again a bundle interpretation can be given, but for the time being we do not further delve into it.

Properties of differential forms

(we shall often omit the adjective "differential")

Given k-forms $\omega_1 = a_I dx^I$, $\omega_2 = b_I dx^I$, their linear combination (with $\alpha, \beta \in \Lambda^0(\mathbb{R}^n)$) is

$$\alpha\omega^1 + \beta\omega^2 = (\alpha a_I + \beta b_I)\, dx^I$$

(as in the algebraic case: everything is carried out pointwise)

Notice that $\mathfrak{X}(\mathbb{R}^n)$ (vector fields) and $\Lambda^k(\mathbb{R}^n)$ are in fact modules over $\Lambda^0(\mathbb{R}^n)$
$X \in \mathfrak{X}(\mathbb{R}^n) \Rightarrow fX \in \mathfrak{X}(\mathbb{R}^n)$
$\omega \in \Lambda^k(\mathbb{R}^n) \Rightarrow f\omega \in \Lambda^k(\mathbb{R}^n)$
etc.

The wedge product between ω_1 (k-form) and ω_2 (l-form), $\omega_1 = a_I dx^I$, $\omega_2 = b_J dx^J$ (again defined pointwise) reads:

$$\omega_1 \wedge \omega_2 = a_I b_J \, dx^I \wedge dx^J$$

Let us check that

(a) $\overset{k}{(\omega} \wedge \overset{s}{\varphi}) \wedge \overset{r}{\nu} = \omega \wedge (\varphi \wedge \nu)$, with $\omega = a_I dx^I$, $\varphi = b_J dx^J$, $\nu = c_K dx^K$ (associativity, therefore one can safely write $\omega \wedge \varphi \wedge \nu$ without ambiguity)

(b) $\omega \wedge (\varphi + \nu) = \omega \wedge \varphi + \omega \wedge \nu$ (easy) if $r = s$

(c) $\overset{k}{\omega} \wedge \overset{s}{\varphi} = (-1)^{ks} \overset{s}{\varphi} \wedge \overset{k}{\omega}$ graded commutativity

Proof of (a): $(\omega \wedge \varphi) \wedge \vartheta = (a_I dx^I \wedge b_J dx^J) \wedge c_K dx^K =$

$= (a_I b_J dx^I \wedge dx^J) \wedge (c_K dx^K) = a_I b_J c_K dx^I \wedge dx^J \wedge dx^K$
$= \text{r.h.s.}$

Proof of (c): $\omega \wedge \varphi = a_I b_J \, dx^{i_1} \wedge .. dx^{i_k} \wedge dx^{j_1} \wedge .. dx^{j_s})$

$= - a_I b_J \, dx^{i_1} \wedge \ldots dx^{j_1} \wedge dx^{i_k} \wedge .. dx^{j_s})$

$\Rightarrow$ there are $\underbrace{k + k + .. k}_{s \text{ times}} = k \cdot s$ sign changes before abutting at $\varphi \wedge \omega$, yielding the $(-1)^{ks}$ factor in the r.h.s.

Notice that in general $\omega \wedge \omega \neq 0$

(If $\omega \in \Lambda^k(\mathbb{R}^n)$, k odd, then $\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega$ odd

$= - \omega \wedge \omega \Rightarrow \omega \wedge \omega = 0$).

If k is even, then one has a tautology: $\omega \wedge \omega = \omega \wedge \omega$

Example: In $\mathbb{R}^4$, take $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \in \Lambda^2(\mathbb{R}^4)$

This is an example of symplectic form

Then $\omega \wedge \omega = \underbrace{dx^1 \wedge dx^2 \wedge dx^1 \wedge dx^2}_{= 0} + dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

$+ dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 + \underbrace{dx^3 \wedge dx^4 \wedge dx^3 \wedge dx^4}_{= 0}$
$= dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

$= 2 \underbrace{dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4}_{\text{volume form on } \mathbb{R}^4}$

* Pull-back of differential forms

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}^m$,

and a k-form $\omega \in \Lambda^k(\mathbb{R}^m)$, one can construct

a k-form $f^*\omega \in \Lambda^k(\mathbb{R}^n)$ (pull-back of ω via f)

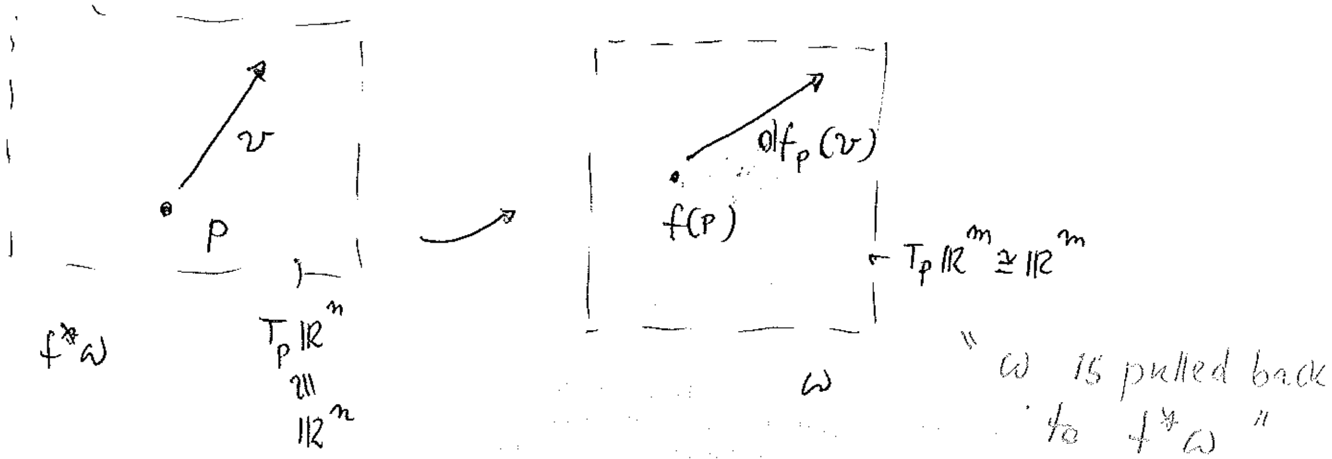
in the following guise:

← pointwise, the k-form is to be evaluated k-tuple of vectors from $T_p\mathbb{R}^n$

$$(f^*\omega)(\underset{\mathbb{R}^n}{p})(\underbrace{v_1 \ldots v_k}_{\in\, T_p\mathbb{R}^n}) := $$

differential of f

$$\omega(f(p))(\underbrace{df_p(v_1), \ldots df_p(v_k)}_{\in\, T_{f(p)}\mathbb{R}^m})$$



If $g \in \Lambda^0(\mathbb{R}^m)$ (a smooth function $g : \mathbb{R}^m \to \mathbb{R}$)

set $f^*g := g \circ f$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

Let us interpret the above formula

Pick $f : \mathbb{R}^n \to \mathbb{R}^m$ (one can in fact work on $\mathcal{U} \subset \mathbb{R}^n$ open)

concretely

$$\begin{cases} y^1 = f^1(x^1 \dots x^n) \\ y^2 = f^2(x^1 \dots x^n) \\ \vdots \\ y^m = f^m(x^1 \dots x^n) \end{cases} \qquad y = f(x)$$

Now:

$$(f^* dy^i)(v) \underset{\text{def}}{=} dy^i(df(v)) \underset{\substack{\text{chain}\\ \text{rule}}}{=} d(y^i \circ f)(v)$$

def of pull-back

$$= d(f^* y^i)(v) = df^i(v)$$

$y^i \circ f = f^i$ ↖ as a function of x

Therefore, operationally, if $\omega = a_I(y) dy^I$,

then $f^*\omega = a_I(f(x)) df^I$

$$dy^I = \frac{\partial y^I}{\partial x^J} dx^J$$ ↖ "partial Jacobians"

If $I = i$, $J = j$ (single indices) $\left(\frac{\partial y^i}{\partial x^j}\right)$ is the Jacobian matrix of df

$$\omega = a_I dy^I \longmapsto f^*\omega = \underbrace{a_I \frac{\partial y^I}{\partial x^J}}_{= a'_J} dx^J$$

See how practical tensor notation is!

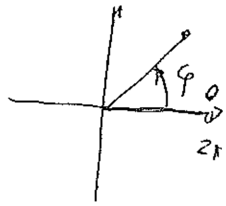
See also below

Example (extremely important)

1. $$\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$ (in $\mathcal{U} = \{ r > 0,\ 0 < \varphi < 2\pi \}$ polar coordinates

defined for $(x,y) \neq (0,0)$

Let $f : \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $\quad x^2+y^2 = r^2$



Then $f^*\omega = \dots = d\varphi$ angular form $\quad \varphi = \arctan\frac{y}{x} \quad x \neq 0$

It is instructive to compute this directly

$$dx = dr\cos\varphi + r\,d\cos\varphi = \cos\varphi\,dr - r\sin\varphi\,d\varphi$$
$$dy = \sin\varphi\,dr + r\cos\varphi\,d\varphi$$

these two terms cancel out

$$\frac{-r\sin\varphi}{r^2}\left(\cos\varphi\,dr - r\sin\varphi\,d\varphi\right) + \frac{r\cos\varphi}{r^2}\left(\sin\varphi\,dr + r\cos\varphi\,d\varphi\right)$$

$$= (\underbrace{\sin^2\varphi + \cos^2\varphi}_{1})\,d\varphi = d\varphi$$

Other examples

Sums over $l_1 \dots l_k$ are omitted

2. $$dy^{i_1}\wedge\dots dy^{i_k} = \frac{\partial y^{i_1}}{\partial x^{l_1}}\,dx^{l_1}\wedge\dots\frac{\partial y^{i_k}}{\partial x^{l_k}}\,dx^{l_k}$$

$$= \underbrace{\frac{\partial(y^{i_1}\dots y^{i_k})}{\partial(x^{l_1}\dots x^{l_k})}}_{\text{"partial Jacobians}}\,dx^{l_1}\wedge\dots dx^{l_k}$$

This is clear from the very definition of determinant involving the appropriate sums over permutations, weighted with $(-1)^\sigma$.
Whenever two equal dx appear, one gets zero by skew-symmetry

Take, for instance

2'. $f : \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

$\omega = dx \wedge dy$ $\qquad f^*\omega = \dfrac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv$

"area 2-form" (oriented)

↖ without $|\ \ |$: one has an oriented area element

Check this directly:

$$\left[\quad \begin{aligned} dx &= x_u\, du + x_v\, dv \\ dy &= y_u\, du + y_v\, dv \end{aligned} \qquad dx \wedge dy = \underbrace{(x_u y_v - x_v y_u)}_{\frac{\partial(x,y)}{\partial(u,v)}}\, du \wedge dv \right]$$

we have omitted the symbol f^*

2''. $f : \mathbb{R}^3 \to \mathbb{R}^2$ $\qquad dy^1 \wedge dy^2 = \dfrac{\partial y^1}{\partial x^{l_1}} \dfrac{\partial y^2}{\partial x^{l_2}}\, dx^{l_1} \wedge dx^{l_2}$

$$= \underbrace{\left(\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} - \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1} \right)}_{\frac{\partial(y^1,y^2)}{\partial(x^1,x^2)}} dx^1 \wedge dx^2 \quad + \text{ similar terms}$$

$$\frac{\partial(y^1,y^2)}{\partial(x^1,x^2)} = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} \end{vmatrix}$$

3. $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ $\qquad \varphi \equiv \underline{r}$

$(u,v) \mapsto (x,y,z)$

F : flux 2-form $\quad (\in \Lambda^2(\mathbb{R}^3))$

$F = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$

$$\varphi^* F = \Big[F_1 (\overbrace{x(u,v), y(u,v), z(u,v)}^{\varphi(u,v)}) \frac{\partial(y,z)}{\partial(u,v)} + F_2 \frac{\partial(z,x)}{\partial(u,v)} + F_3 \frac{\partial(x,y)}{\partial(u,v)} \Big]\, du \wedge dv = \underline{F} \cdot d\underline{\sigma}$$

$d\underline{\sigma} = \underline{n}\, d\sigma$ area element

$$= \langle \underline{F}, \underline{r}_u \times \underline{r}_v \rangle\, du \wedge dv$$

$$\langle \underline{F}, \underline{r}_u \times \underline{r}_v \rangle = \det(\underline{F}, \underline{r}_u, \underline{r}_v)$$

* Properties of pull-back ("functoriality")
compatibility with the various operations

(a) $f^*(\omega+\varphi) = f^*\omega + f^*\varphi$ (easy)

$\nearrow$ k-forms

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}$

(b) $f^*(g\cdot\omega) = f^*(g)\, f^*\omega$ (easy)

$\Lambda^0 \ni g$, $\quad f^*(g) = g\circ f$

1 forms

(c) $f^*(\varphi_1\wedge\varphi_2\wedge\dots\varphi_k) = f^*\varphi_1\wedge f^*\varphi_2\wedge\dots f^*\varphi_k$

this will hold in general

Let us prove it: $f^*(\varphi_1\wedge\dots\varphi_k)(v_1\dots v_k) =$

evaluated at p

$= (\varphi_1\wedge\dots\varphi_k)(df(v_1),\dots df(v_k))$

$=$ $\det(\varphi_i(df(v_j))) = \det(f^*\varphi_i(v_j))$

recall!

$= (f^*\varphi_1\wedge\dots f^*\varphi_k)(v_1,\dots v_k)$ $\square$

(d) in general: $f^*(\omega\wedge\varphi) = f^*\omega\wedge f^*\varphi$

Proof. $\omega = a_I\, dy^I$, $\varphi = b_J\, dy^J$

$f^*(\omega\wedge\varphi) = f^*(a_I b_J\, dy^I\wedge dy^J) = a_I(f^1..f^m)\, b_J(f^1..f^m)\, df^I\wedge df^J =$

⚠ (d) is true for products of 1-forms, see (c)

$= f^*\omega\wedge f^*\varphi$

$\square$

recall the operational interpretation VII-8

(e) $\quad (f\circ g)^* \omega \; = \; g^*(f^*\omega)$ two step pulling-back

direct pulling-back $\qquad$ (beware!) $\qquad \mathbb{R}^p \xrightarrow{g} \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \xleftarrow{g^*} \quad \xleftarrow{f^*}$

$$(f\circ g)^* \omega = a_I\left((f\circ g)^I\right) d(f\circ g)^I$$

chain rule: $d(f\circ g) = df \circ dg$

$$= a_I\left(f^1(g^1\ldots g^n)\ldots f^m(g^1\ldots g^n)\right) \times df^I(dg^1\ldots dg^n)$$

$$= g^*(f^*\omega)$$

just an application of the chain rule

$\square$