Analisi Matematica II - 19 febbraio 2014

Esercizio 3 iii)

Scrivo le equazioni della curva in forma cartesiana parametrica

$$\begin{cases} x = \rho\cos\vartheta \\ y = \rho\sin\vartheta \end{cases} \Rightarrow \varphi: \begin{cases} x = \vartheta\cos\vartheta \\ y = \vartheta\sin\vartheta \end{cases} \qquad \vartheta \in \mathbb{R}^+$$

Derivo

$$\varphi': \begin{cases} x' = \cos\vartheta - \vartheta\sin\vartheta \\ y' = \sin\vartheta + \vartheta\cos\vartheta \end{cases} \Rightarrow \|\varphi(\vartheta)\| = \sqrt{1+\vartheta^2}$$

La lunghezza della curva in funzione di ϑ e'

$$l(\gamma) = \int_0^{2\pi} \|\varphi'(\vartheta)\|\, d\vartheta = \int_0^{2\pi} \sqrt{1+\vartheta^2}\, d\vartheta$$

Risolvo $\int\sqrt{1+\vartheta^2}\, d\vartheta$ mediante la sostituzione $\vartheta = \sinh t$ $\quad d\vartheta = \cosh t\, dt$

$$\int\sqrt{1+\vartheta^2}\, d\vartheta = \int\sqrt{1+\sinh^2 t}\, \cosh t\, dt \Big|_{t=\text{sett}\sinh\vartheta} = \int |\cosh t| \cosh t\, dt = \int \cosh^2 t\, dt =$$

$\cosh(2x) = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1$

$$= \int \frac{1}{2}\cosh(2t) + \frac{1}{2}\, dt = \frac{1}{4}\int 2\cosh(2t)\, dt \Big|_{t=\text{Sett}\sinh x} + \frac{1}{2}t \Big|_{t=\text{Sett}\sinh x} =$$

$$= \frac{1}{4}\sinh(2t)\Big|_{t=\text{Sett}\sinh\vartheta} + \frac{1}{2}t\Big|_{t=\text{Sett}\sinh\vartheta} =$$

$$= \frac{1}{4}\, 2\sinh t\cosh t\Big|_{t=\text{Sett}\sinh\vartheta} + \frac{1}{2}\,\text{Sett}\sinh\vartheta =$$

$$= \frac{1}{2}\sinh t\sqrt{1+\sinh^2 t}\Big|_{t=\text{Sett}\sinh\vartheta} + \frac{1}{2}\,\text{Sett}\sinh\vartheta =$$

$$= \frac{1}{2}\vartheta\sqrt{1+\vartheta^2} + \frac{1}{2}\,\text{Sett}\sinh\vartheta$$

$$\Rightarrow l(\gamma) = \frac{1}{2}\left[2\pi\sqrt{4\pi^2+1} + \lg\left(2\pi + \sqrt{4\pi^2+1}\right)\right]$$