

Addendum alla lezione XIV

TOPOLOGIA E  
 GEOMETRIA DIFFERENZIALE  
 (a.a. 2009/10) Prof. M. Spacca  
 Lezione XV

Vediamo ~~più in dettaglio~~

Come interpretare  $\Pi$  come grafico

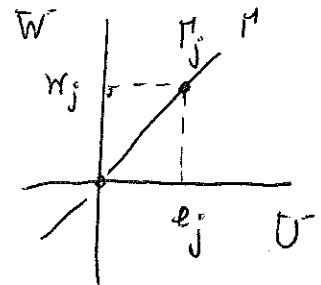
Siano, per fissare le idee  $U = \langle e_1, \dots, e_k \rangle$   
 $\bar{W} = \langle e_{k+1}, \dots, e_n \rangle$

e  $\boxed{\Pi \cap \bar{W} = \{0\}}$

Applicando l'algoritmo di Gauss ad una base di

$\Pi$ , giungiamo a

$$\left( \begin{array}{cccc} \overbrace{1 \ 0 \ 0 \ \dots \ 0}^{n-k} & w_1^{(1)} & \dots & w_{n-k}^{(1)} \\ 0 \ 1 \ 0 \ \dots \ 0 & w_1^{(2)} & \dots & w_{n-k}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ & w_1^{(k)} & \dots & w_{n-k}^{(k)} \end{array} \right) \begin{array}{l} = \Pi_k \\ = \Pi_2 \\ = \Pi_{n-k} \end{array}$$



Si ha  $a_j$  per  $j=1 \dots n-k$

$$U \ni e_j \longmapsto \Pi_j \longmapsto w_j = (0, 0, \dots, 0, \overbrace{w_1^{(1)}, \dots, w_{n-k}^{(1)}}^{n-k}) \in W$$

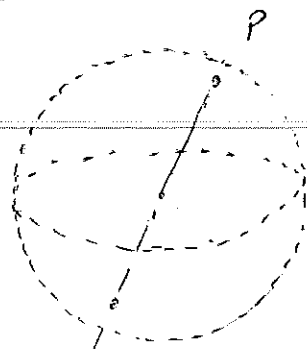
i.e.,  $\Pi$  diviene grafico di  $T: U \rightarrow W$

$$j=1 \dots n-k \quad e_j \xrightarrow{T} w_j$$

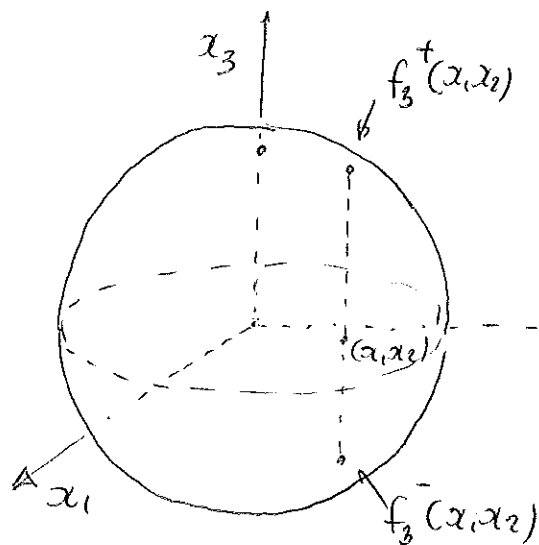
★ un'altra struttura differenziabile su  $\mathbb{P}^2(\mathbb{R})$  (che dà vita allo stesso atlante massimale)

$$\pi: S^2 \rightarrow \mathbb{P}^2(\mathbb{R}) = S^2/a$$

$$p \mapsto \{p, -p\} \equiv [p]$$



Attenzione alla def  $f_\alpha: U_\alpha \rightarrow S^2$  [V. deCarro]



$$f_3^\pm: U_3 \rightarrow S^2$$

$$\{0 \leq x_1^2 + x_2^2 < 1\}$$

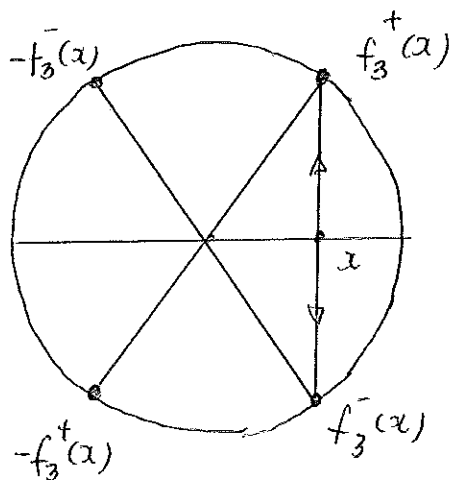
$$(x_1, x_2) \mapsto (x_1, x_2, \pm \sqrt{1 - (x_1^2 + x_2^2)})$$

$D(x_1, x_2)$

$$\pi \circ f_3^+(U_3)$$

$$= \pi \circ f_3^-(U_3)$$

insiemisticamente!  
as sets!



$f_i^\pm$  ... definite analogamente

Sia  $g_i: U_i \rightarrow \mathbb{P}^2(\mathbb{R})$

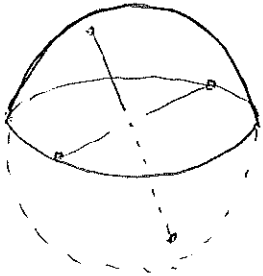
$$g_i = \pi \circ f_i^\pm$$

$\pi|_{f_i^\pm(U_i)}$  è iniettiva (chiaro), pertanto

$$g_i^{-1} \circ g_j = (\pi \circ f_i^+)^{-1} (\pi \circ f_j^+)$$

$$= (f_i^+)^{-1} \circ f_j^+$$

che è  $\mathcal{E}^0$

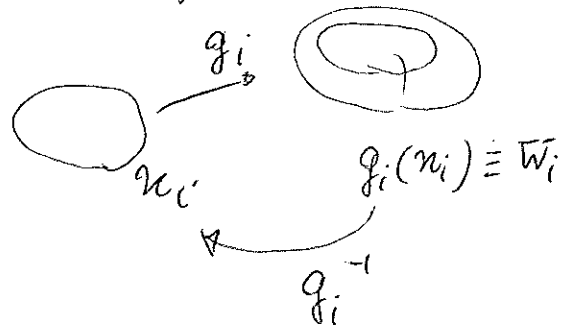


$\Rightarrow \{(U_i, g_i)\}$  struttura diff. pa  $\mathbb{R}^2(\mathbb{R})$

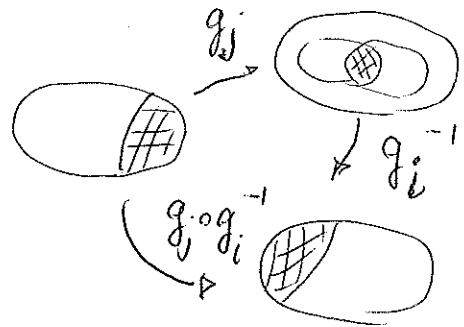


attenzione : con la def.

$$\text{usate } \left\{ \begin{array}{l} (W_i, g_i^{-1}) \\ \cap \\ \mathbb{R}^2(\mathbb{R}) \end{array} \right\} \begin{array}{l} \text{III} \\ g_i \\ \text{Carte} \end{array}$$



III in ogni caso i combiamenti di carte vanno  
comunque da  $\mathbb{R}^n$  in  $\mathbb{R}^n$

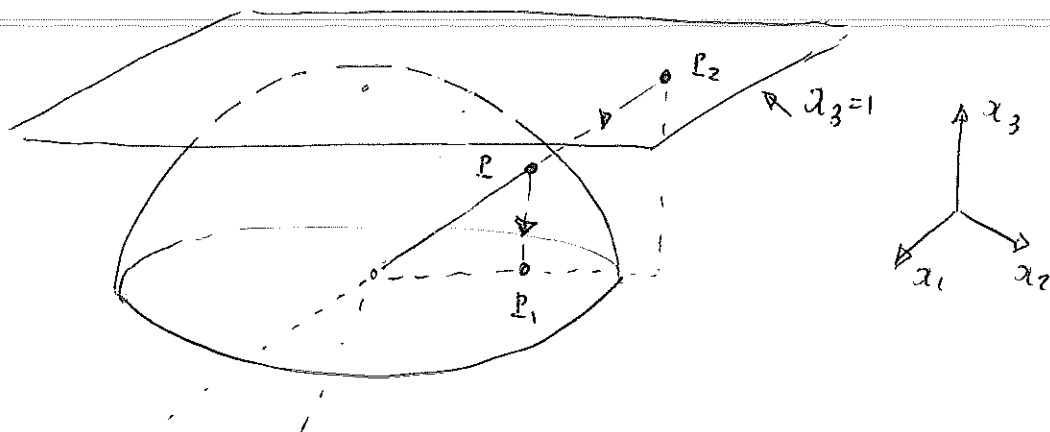


★ compatibilità con l'altro atlante (corno)

$$\left( \begin{array}{l} \frac{x_1}{x_3}, \frac{x_2}{x_3} \end{array} \right) \longleftrightarrow (x_1, x_2)$$

$$\text{qui } x_3 \neq 0 \quad \parallel \quad \sqrt{1-x_1^2-x_2^2} \quad \uparrow \quad \mathcal{E}^0$$

vediamolo in modo ancora più geometrico,  
limitandoci a due carte particolari:



$$P: (x_1, x_2, \sqrt{1-x_1^2-x_2^2}) \xrightarrow{\varphi} (x_1, x_2) = P_1$$

$$\downarrow \psi$$

$$P_2: \left( \frac{x_1}{\sqrt{1-x_1^2-x_2^2}}, \frac{x_2}{\sqrt{1-x_1^2-x_2^2}}, 1 \right)$$

$$\left( \frac{x_1}{\sqrt{1-x_1^2-x_2^2}}, \frac{x_2}{\sqrt{1-x_1^2-x_2^2}} \right)$$

$$\psi \circ \varphi^{-1}: (x_1, x_2) \longmapsto \left( \frac{x_1}{\sqrt{1-x_1^2-x_2^2}}, \frac{x_2}{\sqrt{1-x_1^2-x_2^2}} \right)$$

che è liscia ecc. ecc.

Digressione! Lo spazio prodotto complesso in Meccanica Quantistica è lo spazio degli stati di un sistema quantomeccanico

\* La sfera di Bloch

importante nel calcolo quantistico

Sia  $\mathcal{H}$  sp. vett. hermitiano,  $\dim_{\mathbb{C}} \mathcal{H} = 2$  [i variamente di Hilbert]

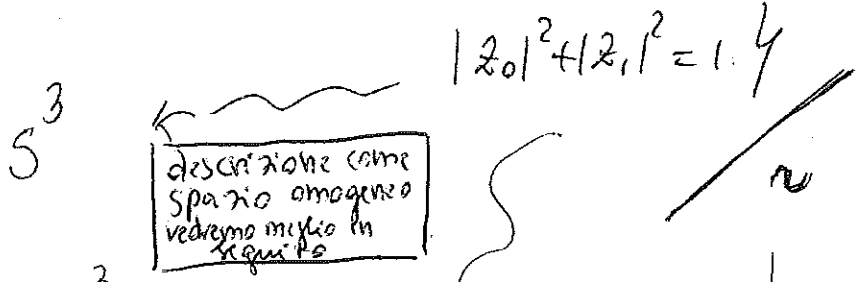
$(|0\rangle, |1\rangle)$  base ortonormale  $\langle , \rangle$  hermitiana  
 \* notazione di Dirac

è una def. equivalente alla precedente

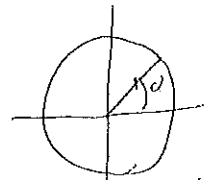
Consideriamo  $\mathbb{P}(\mathcal{H}) \approx \{ \psi \in \mathcal{H} / \|\psi\|=1 \} / \sim$

ovvero  $S^2 \approx \mathbb{P}(\mathbb{C}^2)$  dove  $\psi_1 \sim \psi_2 \Leftrightarrow \psi_2 = e^{i\alpha} \psi_1$

Si ha  $\mathbb{P}(\mathcal{H}) = \{ z_0 |0\rangle + z_1 |1\rangle / \text{fase } |z_0|^2 + |z_1|^2 = 1 \}$



$\mathbb{P}(\mathcal{H}) \approx S^3 / S^1$



Poniamo  $z_0 = \cos(\frac{\vartheta}{2}) |0\rangle$   
 $z_1 = \sin(\frac{\vartheta}{2}) \cdot e^{i\varphi} |1\rangle$

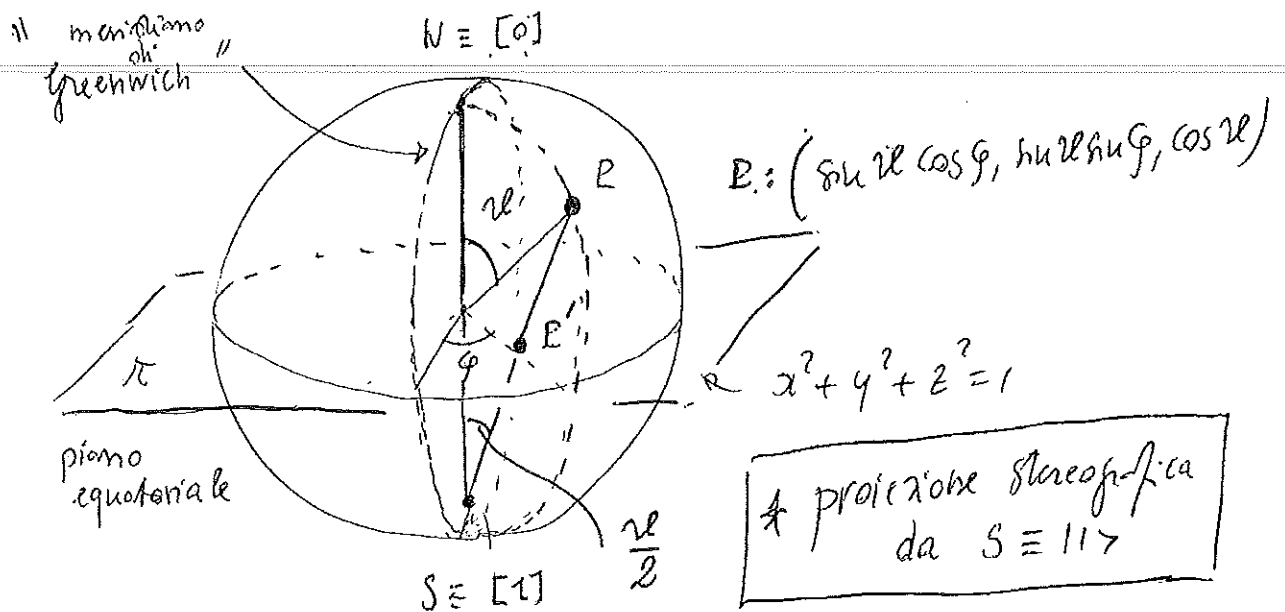
impongo che questo vettore abbia coeff. reale.

$z = \frac{z_1}{z_0} = \tan \frac{\vartheta}{2} e^{i\varphi}$

Nota:  $\mathbb{P}(\mathbb{C}^{n+1}) \approx S^{2n+1} / S^1$   
 (ragionamento analogo)

proiezione stereografica in coordinate sferiche

# ★ la proiezione stereografica in coordinate sferiche



★ retta  $SP$ :

$$\begin{cases} x = t \sin 2\theta \cos \varphi \\ y = t \sin 2\theta \sin \varphi \\ z = -1 + t [\cos 2\theta + 1] \end{cases}$$

$P' = SP \cap \pi \quad z_{P'} = 0 \Rightarrow$

$$t = \frac{1}{1 + \cos 2\theta} \Rightarrow P' = \left( \underbrace{\frac{\sin 2\theta \cos \varphi}{1 + \cos 2\theta}}_x, \underbrace{\frac{\sin 2\theta \sin \varphi}{1 + \cos 2\theta}}_y, 0 \right)$$

ma  $\frac{\sin 2\theta}{1 + \cos 2\theta} = \frac{2 \sin \frac{2\theta}{2} \cos \frac{2\theta}{2}}{2 \cos^2 \frac{2\theta}{2}} = \tan \frac{2\theta}{2}$

$\Rightarrow P' \equiv |\psi\rangle$

$(z = e^{i\varphi} \tan \frac{2\theta}{2})$

$P' = \left( \tan \frac{2\theta}{2} \cos \varphi, \tan \frac{2\theta}{2} \sin \varphi, 0 \right)$

$z = x + iy = e^{i\varphi} \tan \frac{2\theta}{2}$