Theorem 1.1.6 (Definition by Recursion) *Let mappings $H_\square : A^2 \to A$ and $H_\neg : A \to A$ be given and let H_{at} be a mapping from the set of atoms into A, then there exists exactly one mapping $F : PROP \to A$ such that*

$$\begin{cases} F(\varphi) &= H_{at}(\varphi) \text{ for } \varphi \text{ atomic,} \leftarrow \\ F((\varphi \square \psi)) &= H_\square(F(\varphi), F(\psi)), \\ F((\neg\varphi)) &= H_\neg(F(\varphi)). \end{cases}$$

$\alpha \qquad \ell(\alpha) \qquad \ell : PROP \to \mathbb{N}$

$$\begin{bmatrix} \ell[\bot] = 1 \\ \ell[p_i] = 1 \end{bmatrix}$$

$$\ell[(\neg\alpha)] = \ell(\alpha) + 3$$

$$\ell[(\alpha \to \beta)] = \ell(\alpha) + \ell(\beta) + 3$$

Theorem

v: **AT**$\rightarrow \{0, 1\}$ s.t. $v(\perp) = 0$ (assignment for atoms)

$\Rightarrow$

there exists a unique valuation $[\cdot]_v$:**PROP**$\rightarrow$**{0,1}** such that $[\phi]_v = v(\phi)$ for each $\phi \in$**AT**

Definition 2

A mapping v : PROP → {0, 1} is a valuation if

$v(\phi \wedge \psi) = 1 \Leftrightarrow v(\phi)=1$ and $v(\psi)=1$

$v(\phi \vee \psi) = 1 \Leftrightarrow v(\phi)=1$ or $v(\psi)=1$

$v(\phi \rightarrow \psi)=1 \Leftrightarrow v(\phi)=0$ or $v(\psi)=1$,

$v(\phi \leftrightarrow \psi)=1 \Leftrightarrow v(\phi)=v(\psi)$,

$v(\neg\phi) = 1 \Leftrightarrow v(\phi)=0$

$v(\perp) = 0$.

$\alpha \equiv ((p \rightarrow q) \vee (r \wedge s))$

$v : AT \rightarrow \{0, 1\}$ $\quad v(p)=1 \quad v(q)=0 \quad v(r)=1 \quad v(s)=1$

$[\![\alpha]\!]_v = ?$ $\quad [\![((p \rightarrow q) \vee (r \wedge s))]\!]_v = 1 \Leftrightarrow$

1) $[\![(p \rightarrow q)]\!]_v = 1$ OR 2) $[\![(r \wedge s)]\!]_v = 1$

(1) $[\![(p \rightarrow q)]\!]_v = 1 \Leftrightarrow [\![p]\!]_v = 0$ OR $[\![q]\!]_v = 1$

$\Leftrightarrow \underbrace{v(p)=0}_{NO}$ OR $\underbrace{v(q)=1}_{NO}$

quindi $[\![(p \rightarrow q)]\!]_v = 0$

2) $[\![(r \wedge s)]\!]_v = 1$

quindi $\quad [\![\alpha]\!]_v = 1$

Theorem

v: **AT**$\to \{0, 1\}$ s.t. $v(\perp) = 0$ (assignment for atoms)

$\Rightarrow$

there exists a unique valuation $[\cdot]_v$:**PROP**$\to$**{0,1}**
such that $[\phi]_v = v(\phi)$ for each $\phi \in$ **AT**

Definition 2

A mapping v : PROP → {0, 1} is a valuation if

$v(\phi \wedge \psi) = 1 \Leftrightarrow v(\phi)=1$ and $v(\psi)=1$

$v(\phi \vee \psi) = 1 \Leftrightarrow v(\phi)=1$ or $v(\psi)=1$

$v(\phi \to \psi)=1 \Leftrightarrow v(\phi)=0$ or $v(\psi)=1$,

$v(\phi \leftrightarrow \psi)=1 \Leftrightarrow v(\phi)=v(\psi)$,

$v(\neg\phi) = 1 \Leftrightarrow v(\phi)=0$

$v(\perp) = 0$.

$\alpha \equiv (\beta \to \beta)$

$v : AT \to \{0,1\}$

$\forall v \quad [\![\alpha]\!]_v = 1$

$$[\![(\beta \to \beta)]\!]_v = 1 \iff [\![\beta]\!]_v = 0 \text{ or } [\![\beta]\!]_v = 1$$

VERO

$$P \equiv \underline{\text{PIOVE}}$$

$$[\![\underline{\text{PIOVE}} \vee \neg \underline{\text{PIOVE}}]\!]_v = 1$$

$$\Longleftrightarrow$$

$$[\![\underline{\text{PIOVE}}]\!]_v = 1 \quad \text{OR} \quad [\![\neg \underline{\text{PIOVE}}]\!]_v = 1$$

$$\Longleftrightarrow$$

$$[\![\underline{\text{PIOVE}}]\!]_v = 1 \quad \text{OR} \quad [\![\underline{\text{PIOVE}}]\!]_v = 0$$

$$[\![\alpha \vee \neg \alpha]\!]_v = 1 \qquad \forall v : AT \to \{0, 1\}$$

Definition

- ϕ is a **tautology** if $[\phi]_v = 1$ for all valuations v,
- $\models \phi$ stands for 'ϕ is a tautology',
- let Γ be a set of propositions, $\longleftarrow$

  $\Gamma \models \phi$ iff for all v: ($[\psi]_v = 1$ for all $\psi \in \Gamma$) $\Rightarrow [\phi]_v = 1$.

$\{\alpha, \beta, \delta\} \models \phi \equiv \alpha, \beta, \delta \models \phi$

$\alpha \models \alpha \iff \forall v \left([\![\alpha]\!]_v = 1 \Rightarrow [\![\alpha]\!]_v = 1 \right)$ ok!

$\alpha \wedge \beta \models \alpha \iff \forall v \left([\![\alpha \wedge \beta]\!]_v = 1 \Rightarrow [\![\alpha]\!]_v = 1 \right)$

$\iff \forall v \left(\left([\![\alpha]\!]_v = 1 \;\&\; [\![\beta]\!]_v = 1 \right) \Rightarrow [\![\alpha]\!]_v = 1 \right)$ ok

- let Γ be a set of propositions,
  $\Gamma \models \phi$ iff for all v: $([\psi]_v = 1$ for all $\psi \in \Gamma) \Rightarrow [\phi]_v = 1$.

$\alpha, \neg\alpha \models \beta$

$\Leftrightarrow$

$\forall v \left(\left([\![\alpha]\!]_v = 1 \;\&\; [\![\neg\alpha]\!]_v = 1 \right) \Rightarrow [\![\beta]\!]_v = 1 \right)$

$\Leftrightarrow$

$\forall v \left(\left(\underbrace{[\![\alpha]\!]_v = 1 \;\&\; [\![\alpha]\!]_v = 0}_{\text{FALSO}} \right) \Rightarrow [\![\beta]\!]_v = 1 \right)$ OK

[$P \Rightarrow Q$ $\equiv$
P è false OR Q è vera]

falso $\Rightarrow Q$

$\vDash (p \vee q) \to p$?

NO

$\exists v \ [\![(p \vee q) \to p]\!]_v = 0$

$v(p) = 0 \ \& \ v(q) = 1 \ \Rightarrow [\![(p \vee q) \to p]\!]_v = 0$

$[\![\alpha \to \beta]\!]_v = 0 \Leftrightarrow [\![\alpha]\!]_v = 1 \ \& \ [\![\beta]\!]_v = 0$

$v(p) = 0 \ \& \ v(q) = 2 \ \Rightarrow [\![p \vee q]\!]_v = 1 \ \& \ [\![p]\!]_v = 0 \ \Rightarrow$

$[\![(p \vee q) \to p]\!]_v = 0$

$$\models \alpha \to (\alpha \vee \beta)$$

$$\forall v \quad [\![\alpha \to (\alpha \vee \beta)]\!]_v = 1$$

$$\Leftrightarrow$$

$$\forall v \left([\![\alpha]\!]_v = 0 \ \text{ or } \ [\![\alpha \vee \beta]\!]_v = 1 \right)$$

$$\Leftrightarrow$$

$$\forall v \left([\![\alpha]\!]_v = 0 \ \text{ or } \left([\![\alpha]\!]_v = 1 \ \text{ or } \ [\![\beta]\!]_v = 1 \right) \right)$$

$$\Leftrightarrow$$

$$\forall v \left([\![\alpha]\!]_v = 0 \ \text{ or } \ [\![\alpha]\!]_v = 1 \ \text{ or } \ [\![\beta]\!]_v = 1 \right) \quad \text{ok}$$

$$\not\models p \to (p \wedge q)$$

$$v(p)=1 \qquad v(q)=0 \quad \Rightarrow$$

$$[\![p]\!]_v = 1 \ \& \ [\![p \wedge q]\!]_v = 0 \quad \Rightarrow$$

$$\cdot \ [\![p \to (p \wedge q)]\!]_v = 0$$