2D Discrete Fourier Transform (DFT)

Outline

- Circular and linear convolutions

- 2D DFT

- 2D DCT

- Properties

- Other formulations

- Examples

2D Discrete Fourier Transform

- Fourier transform of a 2D signal defined over a discrete finite 2D grid of size MxN

    or equivalently

- Fourier transform of a 2D set of samples forming a bidimensional sequence

- As in the 1D case, 2D-DFT, though a self-consistent transform, can be considered as a mean of calculating the transform of a 2D sampled signal defined over a discrete grid.

- The signal is periodized along both dimensions and the 2D-DFT can be regarded as a sampled version of the 2D DTFT

2D Discrete Fourier Transform (2D DFT)

- 2D Fourier (discrete time) Transform (DTFT) [Gonzalez]

$$F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f[m,n]e^{-j2\pi(um+vn)}$$

a-periodic signal
periodic transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n]e^{-j2\pi\left(\frac{k}{M}m+\frac{l}{N}n\right)}$$
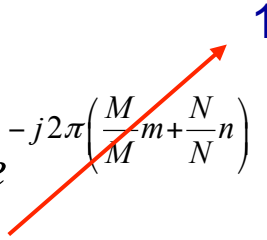
periodized signal
periodic and **sampled**
transform

2D DFT can be regarded as a sampled version of 2D DTFT.

2D DFT: Periodicity

- A [M,N] point DFT is periodic with period [M,N]
  - Proof

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi \left(\frac{k}{M} m + \frac{l}{N} n \right)}$$

$$F[k+M,l+N] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi \left(\frac{k+M}{M} m + \frac{l+N}{N} n \right)}$$

$$= \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi \left(\frac{k}{M} m + \frac{l}{N} n \right)} \underbrace{e^{-j2\pi \left(\frac{M}{M} m + \frac{N}{N} n \right)}}_{1}$$

$$= F[k,l]$$

(In what follows: spatial coordinates=k,l, frequency coordinates: u,v)

2D DFT: Periodicity

- Periodicity

$$F[u,v] = F[u+mM,v] = F[u,v+nN] = F[u+mM,v+nN]$$
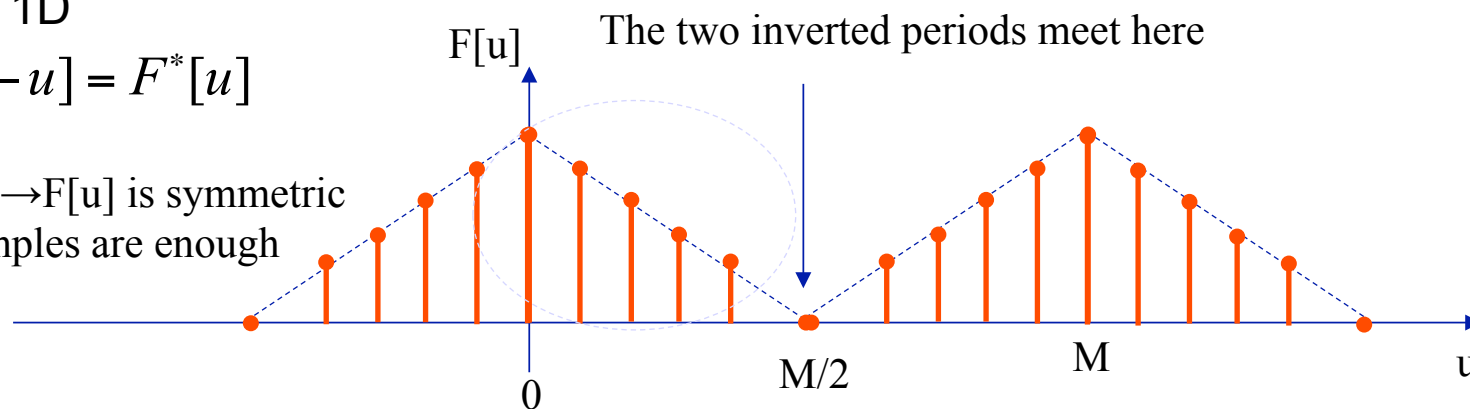
$$f[k,l] = f[k+mM,l] = f[k,l+nN] = f[k+mM,l+nN]$$

- This has important consequences on the implementation and energy compaction property
  - 1D

$$F[N-u] = F^*[u]$$

f[k] real→F[u] is symmetric
M/2 samples are enough

F[u]

The two inverted periods meet here

0   M/2   M   u

Periodicity: 1D

$$f[k] \leftrightarrow F[u]$$

changing the sign of every other sample of the DFT puts F[0] at the center of the interval [0,M]
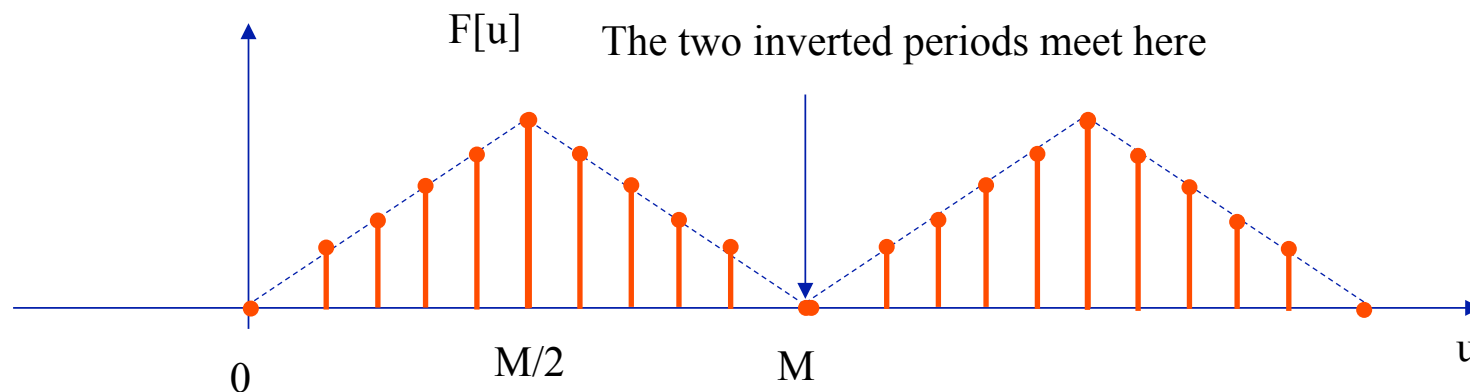
$$f[k] e^{j 2\pi \frac{u_0 k}{M}} \leftrightarrow F[u - u_0]$$

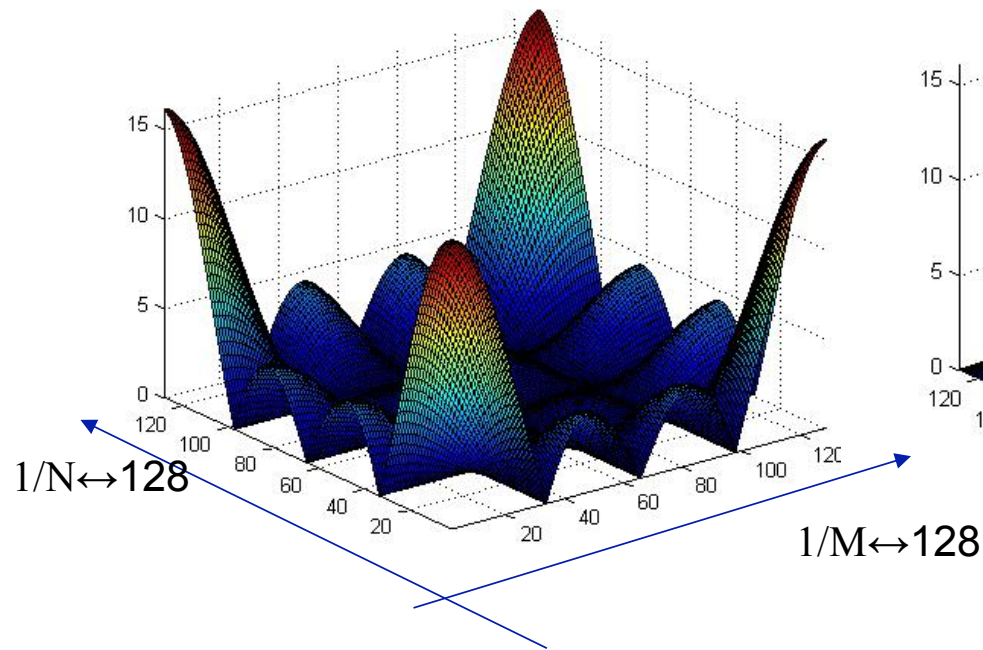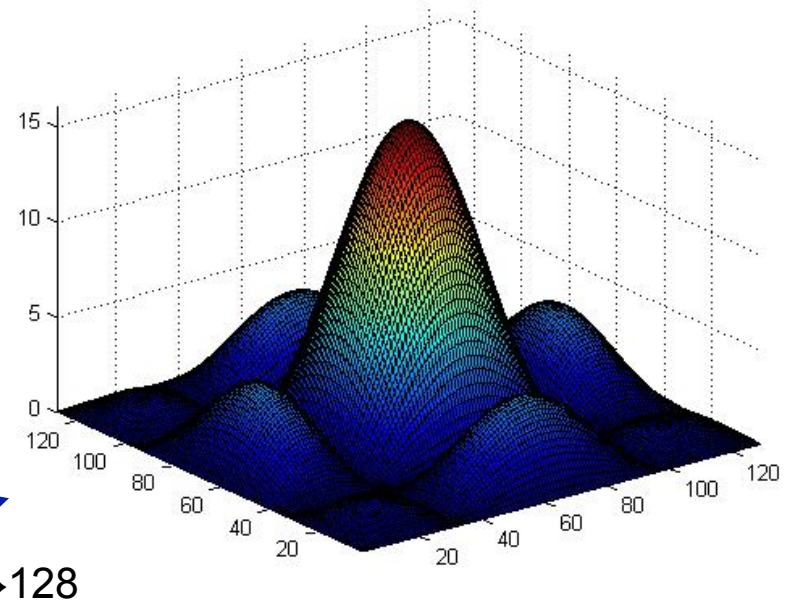$$u_0 = \frac{M}{2} \rightarrow e^{j 2\pi \frac{u_0 k}{M}} = e^{j 2\pi \frac{Mk}{2M}} = e^{j\pi k} = (-1)^k$$

$$(-1)^k f[k] \leftrightarrow F[u - \frac{M}{2}]$$

F[u]

The two inverted periods meet here

0   M/2   M   u

It is more practical to have one complete period positioned in [0, M-1]
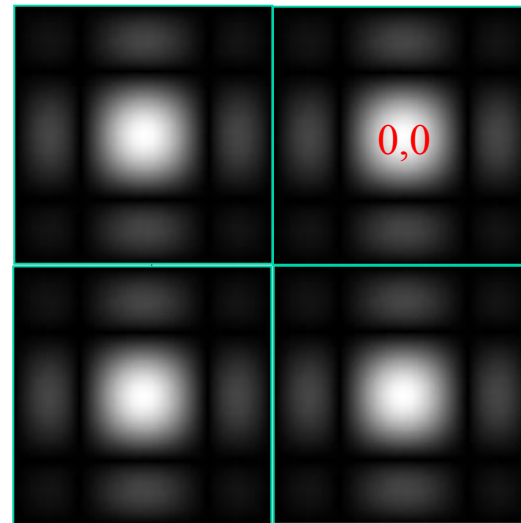
Periodicity in 2D

1/N↔128

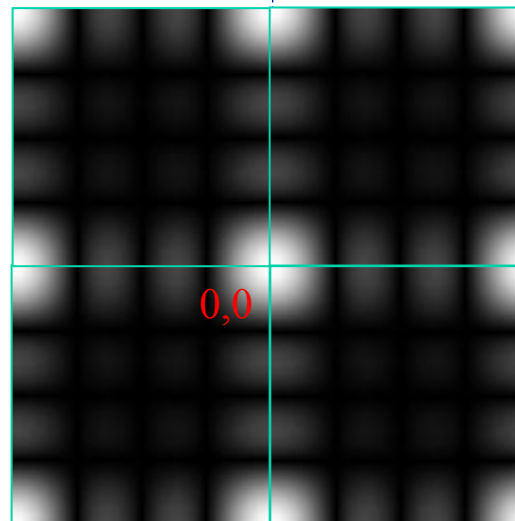1/M↔128

I 4 semiperiodi si incontrano ai vertici

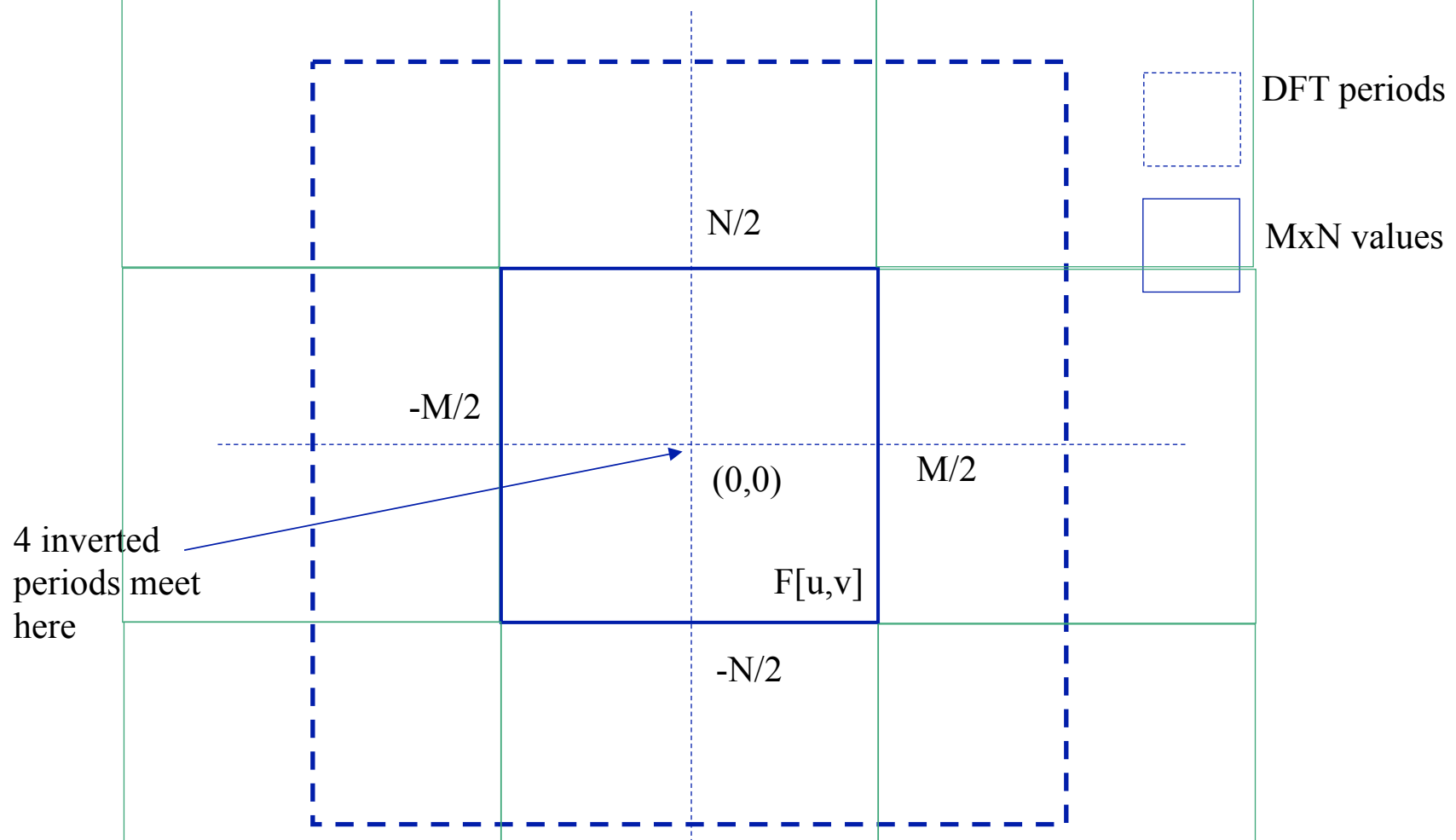I 4 semiperiodi si incontrano al centro

Periodicity

0,0

fftshift(fft2)

0,0   0,127=1/M,1/N

fft2

Periodicity: 2D

DFT periods

MxN values

N/2

-M/2

(0,0)

M/2

4 inverted periods meet here

F[u,v]

-N/2

13

Periodicity: 2D

F[u,v]

data contain one centered
complete period

$$f[k,l]e^{j2\pi(\frac{u_0 k}{M}+\frac{v_0 l}{N})} \leftrightarrow F[u-u_0, v-v_0]$$

$$u_0 = \frac{M}{2}, v_0 = \frac{N}{2} \rightarrow$$

$$(-1)^{k+l} f[k] \leftrightarrow F\left[u-\frac{M}{2}, v-\frac{N}{2}\right]$$

N-1

DFT periods

MxN values

N/2

(0,0)

M/2

M-1

4 inverted
periods meet
here

Periodicity: 2D

F[u,v]

N-1

M/2

(0,0)

M-1

N/2

4 inverted
periods meet
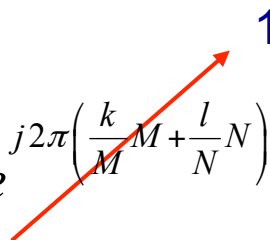here
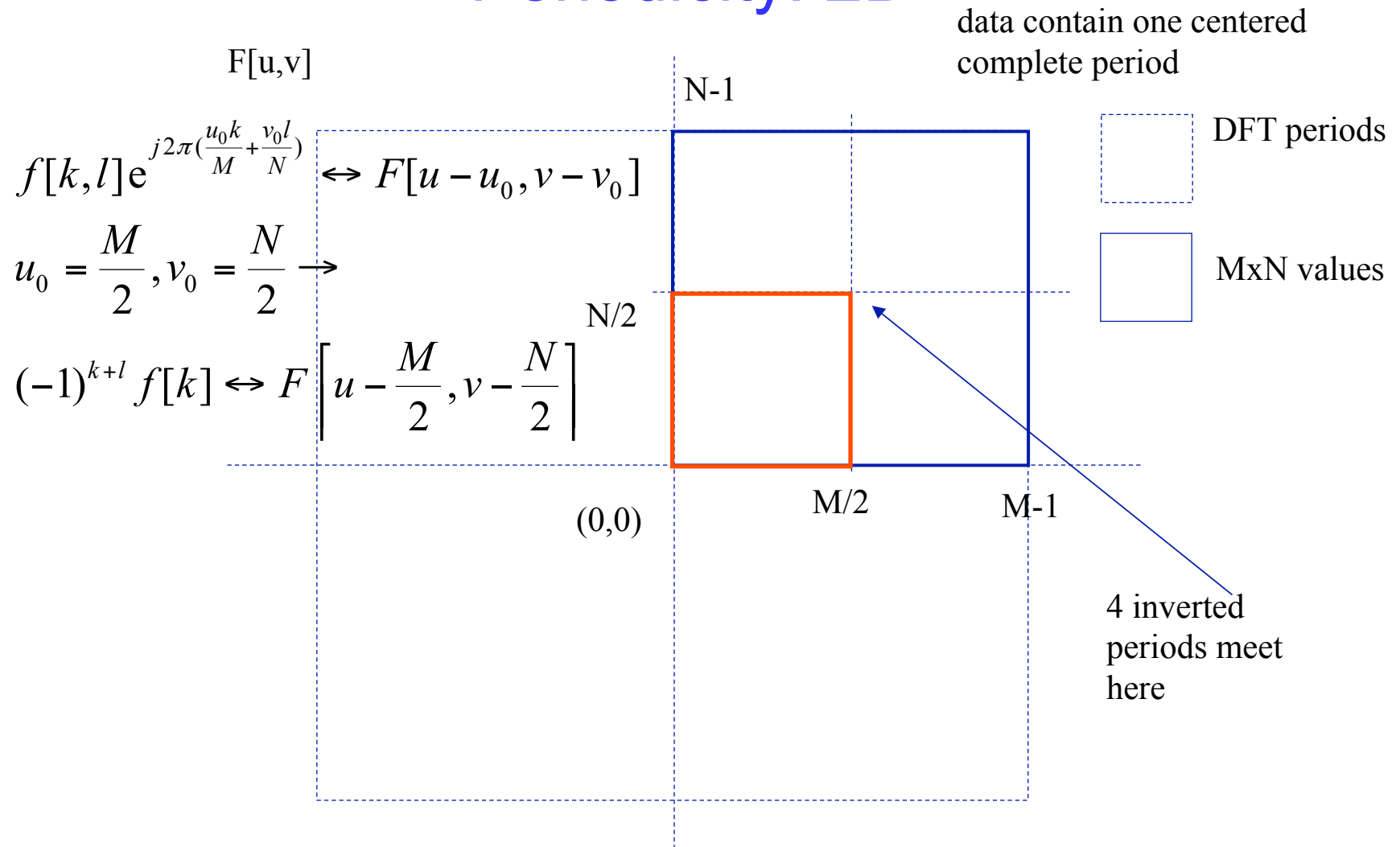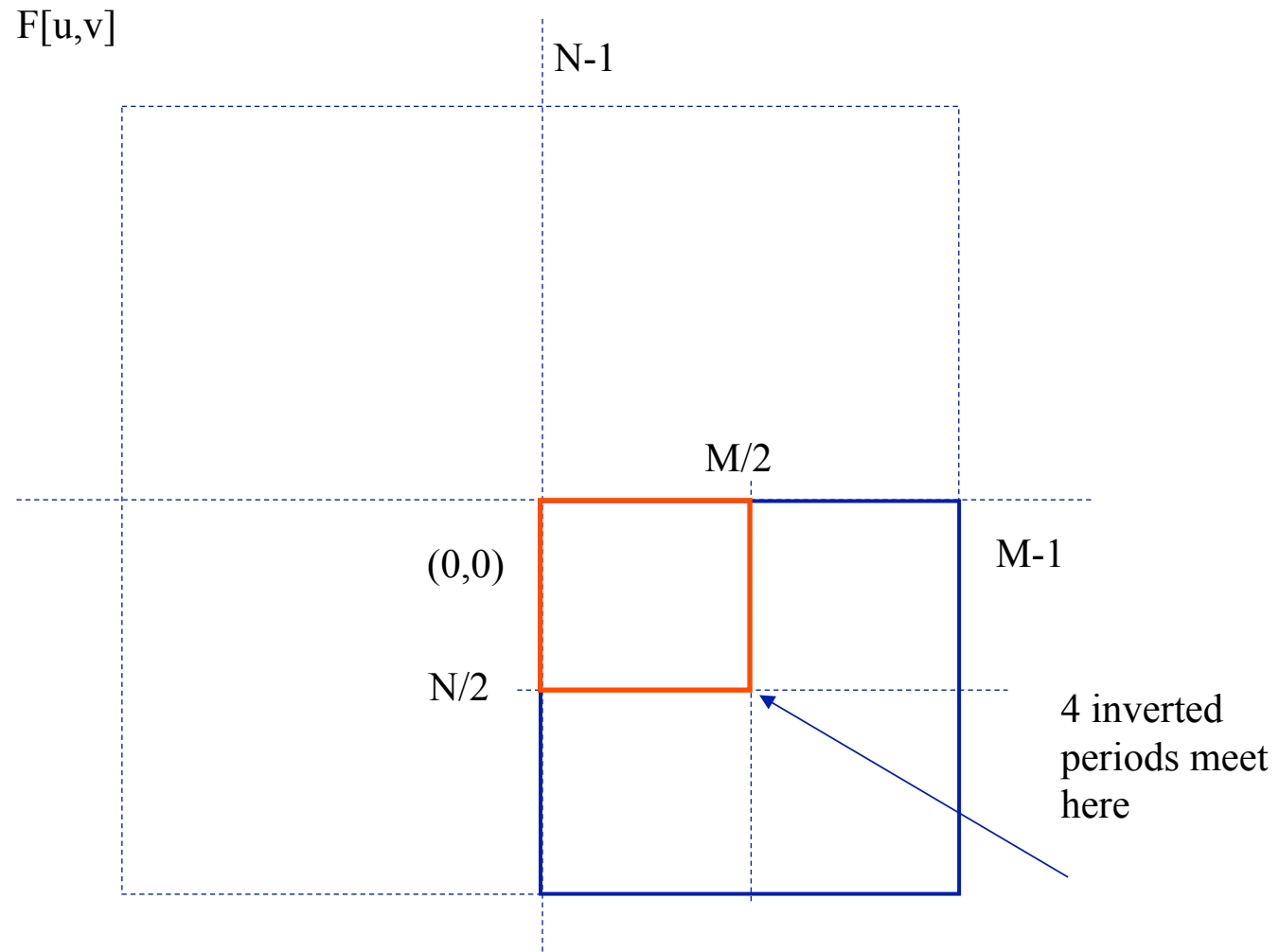
Periodicity in spatial domain

- [M,N] point inverse DFT is periodic with period [M,N]

$$f[m,n] = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

$$f[m+M,n+N] = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}(m+M) + \frac{l}{N}(n+N)\right)}$$

1

$$= \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)} e^{j2\pi\left(\frac{k}{M}M + \frac{l}{N}N\right)}$$

$$= f[m,n]$$

Angle and phase spectra

$$F[u,v] = |F[u,v]| e^{j\Phi[u,v]}$$

$$|F[u,v]| = \left[\mathrm{Re}\{F[u,v]\}^2 + \mathrm{Im}\{F[u,v]\}^2 \right]^{1/2} \qquad \text{modulus (amplitude spectrum)}$$

$$\Phi[u,v] = \arctan\left[\frac{\mathrm{Im}\{F[u,v]\}}{\mathrm{Re}\{F[u,v]\}} \right] \qquad \text{phase}$$

$$P[u,v] = |F[u,v]|^2 \qquad \text{power spectrum}$$

For a real function

$$F[-u,-v] = F^*[u,v] \qquad \text{conjugate symmetric with respect to the origin}$$

$$|F[-u,-v]| = |F[u,v]|$$

$$\Phi[-u,-v] = -\Phi[u,v]$$

Translation and rotation

$$f[k,l]e^{j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)} \leftrightarrow F[u-m,v-l]$$

$$f[k-m,l-n] \leftrightarrow F[u,v]^{-j2\pi\left(\frac{m}{M}k+\frac{n}{N}l\right)}$$

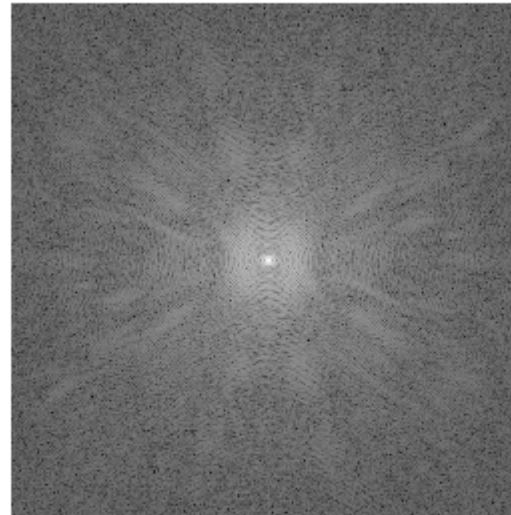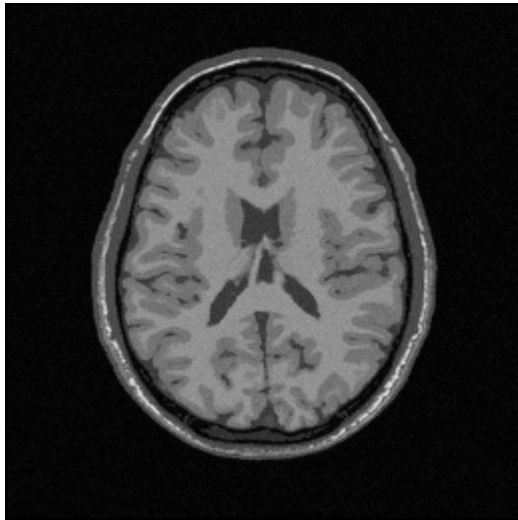$$\begin{cases} k = r\cos\vartheta \\ l = r\sin\vartheta \end{cases} \quad \begin{cases} u = \omega\cos\varphi \\ l = \omega\sin\varphi \end{cases}$$

$$f[r,\vartheta+\vartheta_0] \leftrightarrow F[\omega,\varphi+\vartheta_0]$$

Rotations in spatial domain correspond equal rotations in Fourier domain

mean value

$$F[0,0] = \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} f[n,m]$$

DC coefficient

Separability

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

- inverse transform

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

  - Because the transform kernels are separable and symmetric, the two dimensional transforms can be computed as sequential row and column one-dimensional transforms.
  - The basis functions of the transform are complex exponentials that may be decomposed into sine and cosine components.

2D DFT: summary

TABLE 4.1

Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u, v) = \dfrac{1}{MN} \sum\limits_{x=0}^{M-1} \sum\limits_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$
Inverse Fourier transform	$f(x, y) = \sum\limits_{u=0}^{M-1} \sum\limits_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
Polar representation	$F(u, v) = \|F(u, v)\| e^{-j\phi(u,v)}$
Spectrum	$\|F(u, v)\| = \left[R^2(u, v) + I^2(u, v) \right]^{1/2}, \quad R = \text{Real}(F) \text{ and } I = \text{Imag}(F)$
Phase angle	$\phi(u, v) = \tan^{-1}\left[\dfrac{I(u, v)}{R(u, v)} \right]$
Power spectrum	$P(u, v) = \|F(u, v)\|^2$
Average value	$\bar{f}(x, y) = F(0, 0) = \dfrac{1}{MN} \sum\limits_{x=0}^{M-1} \sum\limits_{y=0}^{N-1} f(x, y)$
Translation	$f(x, y) e^{j2\pi(u_0 x/M + v_0 y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v) e^{-j2\pi(ux_0/M + vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$

2D DFT: summary

Conjugate symmetry	$F(u, v) = F^*(-u, -v)$ $\|F(u, v)\| = \|F(-u, -v)\|$
Differentiation	$\dfrac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$ $(-jx)^n f(x, y) \Leftrightarrow \dfrac{\partial^n F(u, v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2) F(u, v)$
Distributivity	$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$ $\Im[f_1(x, y) \cdot f_2(x, y)] \neq \Im[f_1(x, y)] \cdot \Im[f_2(x, y)]$
Scaling	$af(x, y) \Leftrightarrow aF(u, v), f(ax, by) \Leftrightarrow \dfrac{1}{\|ab\|} F(u/a, v/b)$
Rotation	$x = r \cos\theta \quad y = r \sin\theta \quad u = \omega \cos\varphi \quad v = \omega \sin\varphi$ $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$
Periodicity	$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$ $f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

2D DFT: summary

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$$\frac{1}{MN} f^*(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v) e^{-j2\pi(ux/M + vy/N)}$$ This equation indicates that inputting the function $F^*(u, v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x, y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution[†]	$$f(x, y) * h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n) h(x - m, y - n)$$
Correlation[†]	$$f(x, y) \circ h(x, y) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n) h(x + m, y + n)$$
Convolution theorem[†]	$f(x, y) * h(x, y) \Leftrightarrow F(u, v) H(u, v)$; $f(x, y) h(x, y) \Leftrightarrow F(u, v) * H(u, v)$
Correlation theorem[†]	$f(x, y) \circ h(x, y) \Leftrightarrow F^*(u, v) H(u, v)$; $f^*(x, y) h(x, y) \Leftrightarrow F(u, v) \circ H(u, v)$

2D DFT: summary

Some useful FT pairs:

Impulse $\qquad \delta(x, y) \Leftrightarrow 1$

Gaussian $\qquad A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$

Rectangle $\qquad \text{rect}[a, b] \Leftrightarrow ab\dfrac{\sin(\pi ua)}{(\pi ua)}\dfrac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$

Cosine $\qquad \cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$
$$\frac{1}{2}\left[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)\right]$$

Sine $\qquad \sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$
$$j\frac{1}{2}\left[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)\right]$$

[†] Assumes that functions have been extended by zero padding.

other formulations

2D Discrete Fourier Transform

- 2D Discrete Fourier Transform (DFT)

$$F[k,l] = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $l = 0,1,...,N-1$

$k = 0,1,...,M-1$

- Inverse DFT

$$f[m,n] = \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- It is also possible to define DFT as follows

$$F[k,l] = \frac{1}{\sqrt{MN}} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where
$$k = 0,1,...,M-1$$
$$l = 0,1,...,N-1$$

- Inverse DFT

$$f[m,n] = \frac{1}{\sqrt{MN}} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D Discrete Fourier Transform

- Or, as follows

$$F[k,l] = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f[m,n] e^{-j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

where $k = 0, 1, ..., M-1$ and $l = 0, 1, ..., N-1$

- Inverse DFT

$$f[m,n] = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F[k,l] e^{j2\pi\left(\frac{k}{M}m + \frac{l}{N}n\right)}$$

2D DFT

- The discrete two-dimensional Fourier transform of an image array is defined in series form as

$$\mathcal{F}(u, v) = \frac{1}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} F(j, k) \exp\left\{\frac{-2\pi i}{N}(uj + vk)\right\}$$

- inverse transform

$$F(j, k) = \frac{1}{N} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \mathcal{F}(u, v) \exp\left\{\frac{2\pi i}{N}(uj + vk)\right\}$$

2D DCT

Discrete Cosine Transform

2D DCT

- based on most common form for 1D DCT

$$C(u) = \alpha(u) \sum_{x=0}^{N-1} f(x) \cos\left[\frac{\pi(2x+1)u}{2N}\right];$$ u,x=0,1,…, N-1

$$f(x) = \sum_{u=0}^{N-1} \alpha(u) C(u) \cos\left[\frac{\pi(2x+1)u}{2N}\right],$$

$$\alpha(u) = \begin{cases} \sqrt{\dfrac{1}{N}} & for \quad u = 0 \\[2ex] \sqrt{\dfrac{2}{N}} & for \quad u \neq 0. \end{cases}$$

$$C(u = 0) = \sqrt{\frac{1}{N}} \sum_{x=0}^{N-1} f(x).$$ "mean" value

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1D basis functions

Figure 1
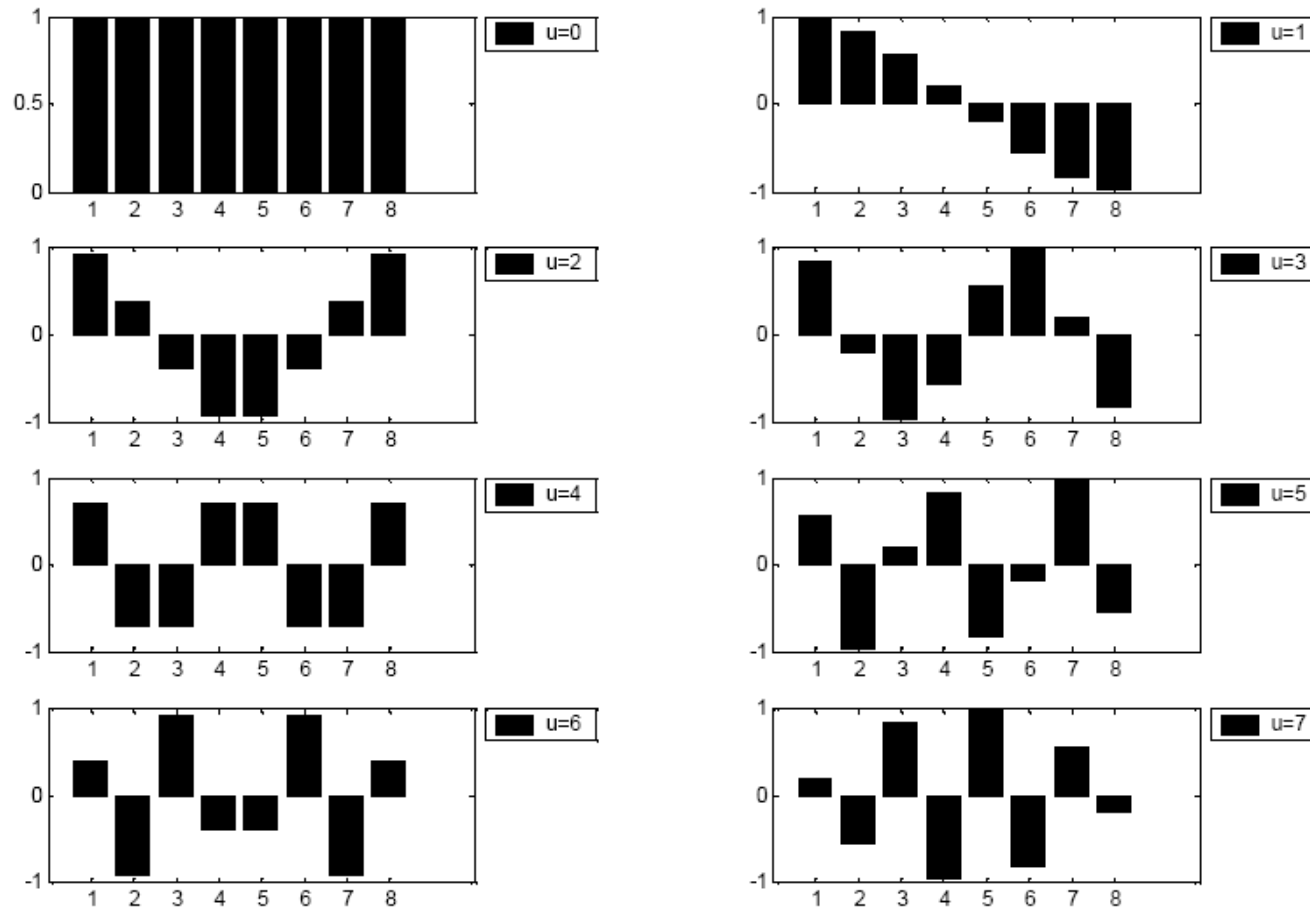
Cosine basis functions are orthogonal

2D DCT

- Corresponding 2D formulation

direct
$$C(u,v) = \alpha(u)\alpha(v) \sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

$u,v = 0, 1, ..., N-1$

$$\alpha(u) = \begin{cases} \sqrt{\dfrac{1}{N}} & \text{for} \quad u = 0 \\ \sqrt{\dfrac{2}{N}} & \text{for} \quad u \neq 0. \end{cases}$$

inverse
$$f(x,y) = \sum_{u=0}^{N-1}\sum_{v=0}^{N-1} \alpha(u)\alpha(v)C(u,v)\cos\left[\frac{\pi(2x+1)u}{2N}\right]\cos\left[\frac{\pi(2y+1)v}{2N}\right],$$

2D basis functions

- The 2-D basis functions can be generated by multiplying the horizontally oriented 1-D basis functions (shown in Figure 1) with vertically oriented set of the same functions.

- The basis functions for $N = 8$ are shown in Figure 2.
  - The basis functions exhibit a progressive increase in frequency both in the vertical and horizontal direction.
  - The top left basis function assumes a constant value and is referred to as the *DC coefficient*.

2D DCT basis functions

Figure 2
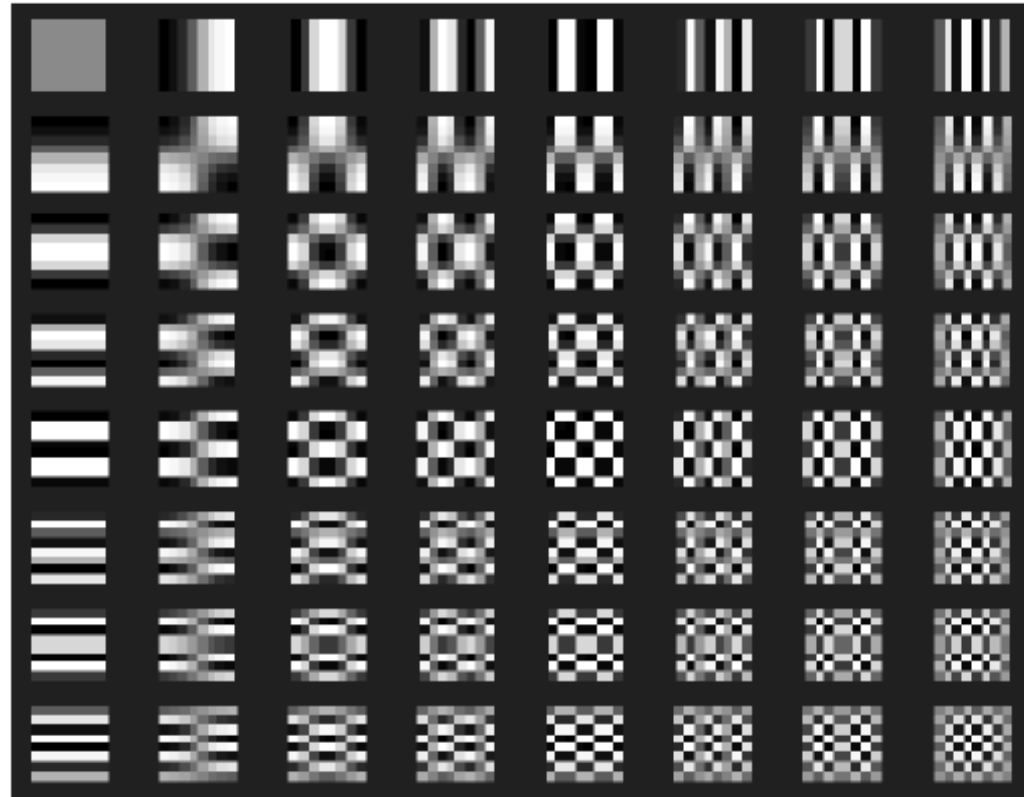
Separability
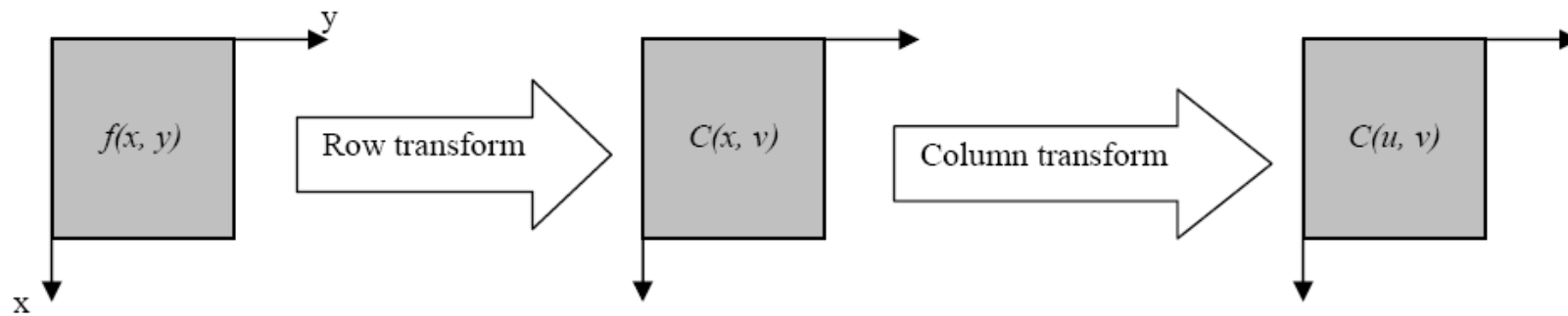
The inverse of a multi-dimensional DCT is just a separable product of the inverse(s) of the corresponding one-dimensional DCT , e.g. the one-dimensional inverses applied along one dimension at a time

Block-based implementation

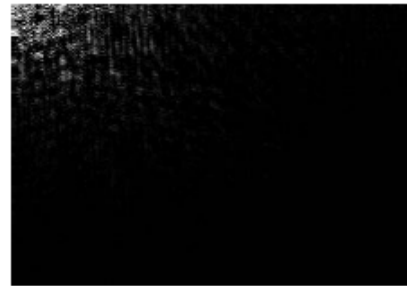Basis function

Block-based transform

Block size
N=M=8

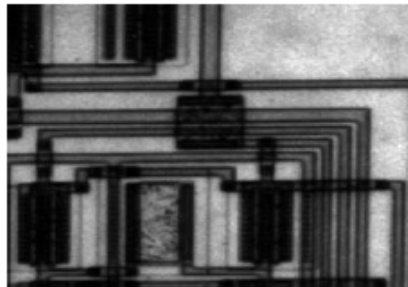The source data (8x8) is transformed to a linear combination of these 64 frequency squares.
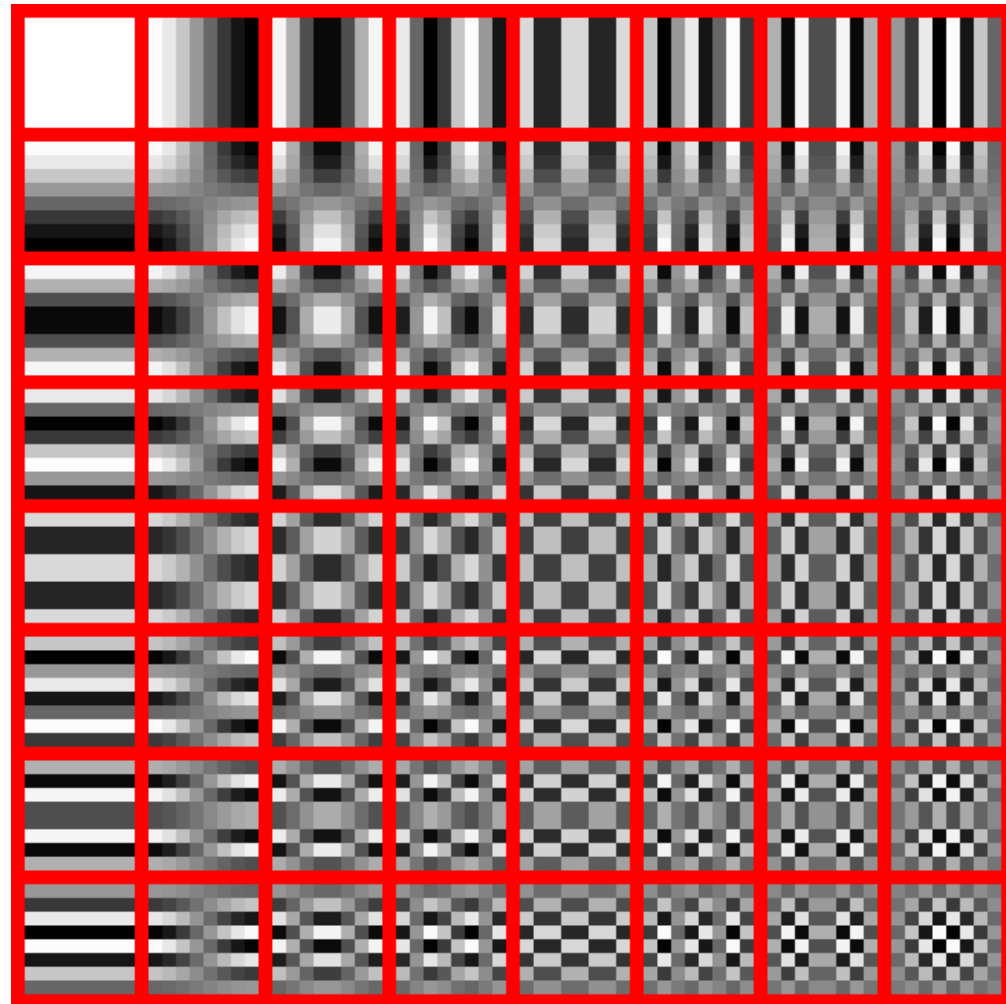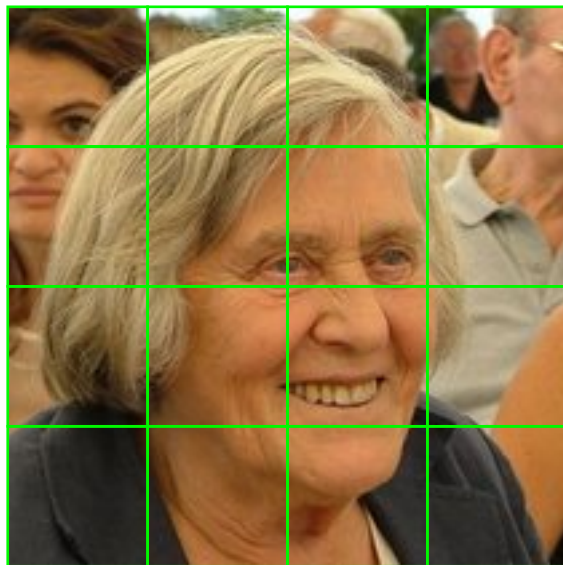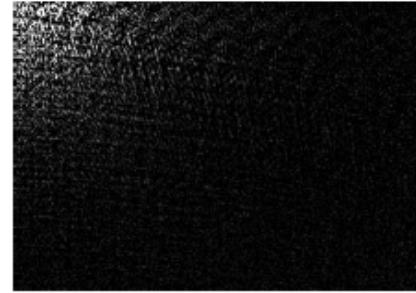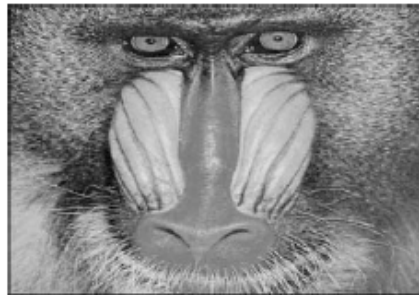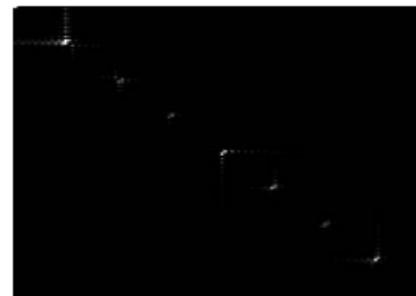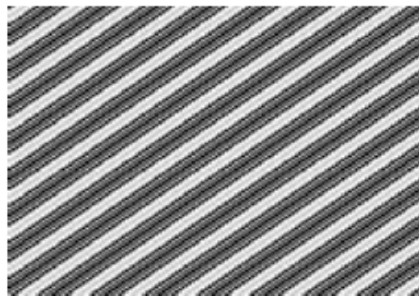
Energy compaction
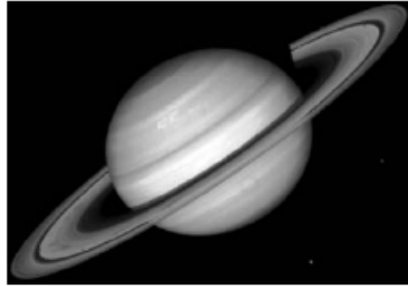
(a)

(b)

Energy compaction

(d)

(e)

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Appendix

- Eulero's formula

$$A(j, k\, ;\, u, v) = \exp\left\{\frac{-2\pi i}{N}(uj + vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj + vk)\right\} - i\sin\left\{\frac{2\pi}{N}(uj + vk)\right\}$$

$$B(j, k\, ;\, u, v) = \exp\left\{\frac{2\pi i}{N}(uj + vk)\right\} = \cos\left\{\frac{2\pi}{N}(uj + vk)\right\} + i\sin\left\{\frac{2\pi}{N}(uj + vk)\right\}$$