— Lectures on —

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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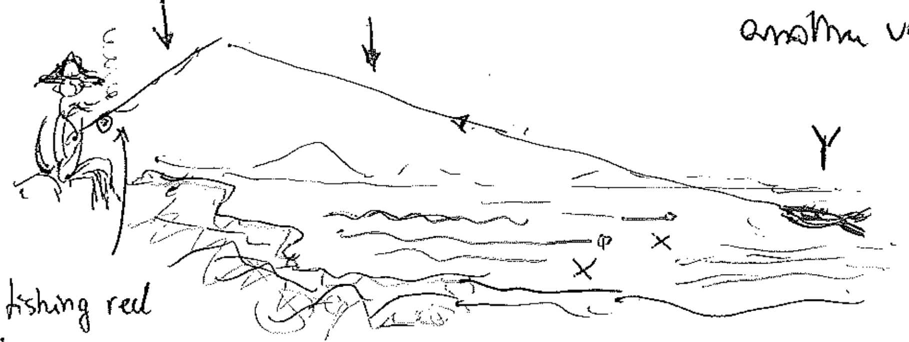
Lecture XVI

* Lie derivative of a vector field

Fisherman's derivative

It differentiates a vector field Y on M, along the integral curves of another vector field X on M and produces another vector field, denoted by $\mathcal{L}_X Y$.



Here is the definition:

$$(\mathcal{L}_X Y)(p) := \frac{d}{dt}\underbrace{\left[\left(F_t^X\right)^{-1}\right]_* Y(F_t^X\cdot p)}_{\in\, T_pM}\Bigg|_{t=0}$$

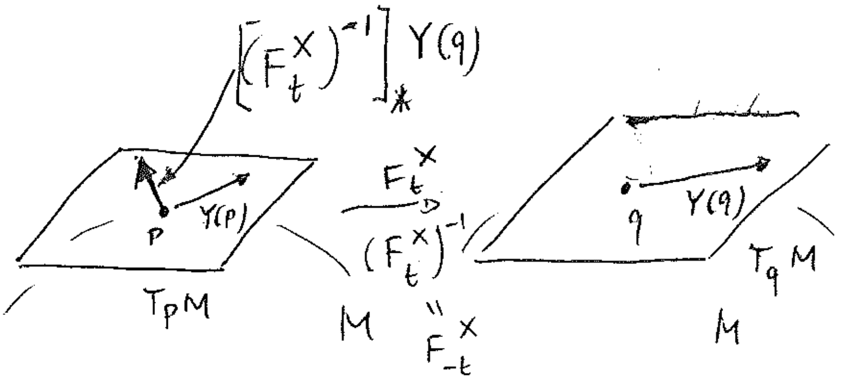
$\boxed{(F^{-1})_* = F_*^{-1}}$

$$= \frac{d}{dt}\left(F_{-t}^X\right)_* Y(F_t^X\cdot p)\Big|_{t=0}$$

$\mathcal{L}_X Y$: Lie derivative of Y along X

or, equivalently: $$(\mathcal{L}_X Y)(p) = \lim_{t\to 0}\frac{\left[\left(F_t^X\right)^{-1}\right]_* Y(F_t^X\cdot p) - Y(p)}{t}$$

they live in the same space.



$q = F_t^X\cdot p$

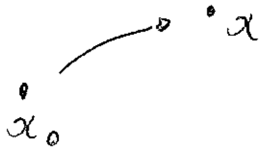
* **Theorem** $\forall X, Y \in \mathcal{X}(M)$

$$\mathcal{L}_X Y = [X, Y] \qquad \text{(Lie bracket)}$$

First proof : use local coordinates

$$X \rightsquigarrow \xi = (\xi^i) \qquad X = \xi^i \partial_i$$

$$Y \rightsquigarrow \eta = (\eta^i) \qquad Y = \eta^i \partial_i$$



$$F_t^X : \quad x = x_0 + t\xi + ..$$

$$x^i(t, x_0^1 .. x_0^n) = x_0^i + t\xi^i(x_0^1 .. x_0^n) + o(t)$$

$$x_0 = x - t\xi + ..$$

$$(F_t^{-1})_* = I - t\frac{\partial \xi}{\partial x} \quad \text{in components: } \delta^i_j - t\frac{\partial \xi^i}{\partial x^j} \quad \boxed{(F_0^{-1})_* = I}$$

$$((F_t^{-1})_* \eta)(x_0) = \eta^j(x)\frac{\partial x_0^i}{\partial x^j} \quad \leftarrow \text{notice that both } F \text{ and } \eta \text{ depend on } t$$

Let us differentiate at $t = 0$ (i.e. at x_0)

$$\frac{d}{dt}\left[(F_t^{-1})_* \eta\right]_{t=0} = \left[\frac{d(F_t^{-1})_*}{dt}\eta + (F_t^{-1})_*\frac{d\eta}{dt}\right]_{t=0}$$

$$= \left(-\eta^j\frac{\partial \xi^i}{\partial x^j} + \frac{\partial \eta^i}{\partial x^j}\frac{dx^j}{dt}\right)_{t=0} = -\eta^j\frac{\partial \xi^i}{\partial x^j} + \xi^j\frac{\partial \eta^i}{\partial x^j}$$

$$= [\xi, \eta] \rightsquigarrow [X, Y].$$

This concludes the proof. $\square$

$$\boxed{\mathcal{L}_X Y = [X,Y]}$$ "intrinsic" proof

Start from $(\varphi_* v)(f)(\varphi(p)) = v(f\circ\varphi)(p)$

or, equivalently $\boxed{(\varphi_* v)(f)(q) = v(f\circ\varphi)(\varphi^{-1}(q))}$

Now let $\varphi = \varphi^X_{-t}$ (flow of X). Then (use this)

$$((\varphi^X_{-t})_* Y)(f)(p) = Y(f\circ\varphi^X_{-t})(\varphi^X_t(p))$$

$$(\mathcal{L}_X Y)(f)(p) = \lim_{t\to 0} \frac{\left[(\varphi^X_{-t})_* Y\right](f)(p) - (Yf)(p)}{t}$$

$$= \lim_{t\to 0} \frac{Y(f\circ\varphi^X_{-t})(\varphi^X_t(p)) - (Yf)(p)}{t} \qquad (\diamond)$$

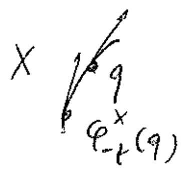
Willmore

$\star$ Let $\quad f(q) - f(\varphi^X_{-t}(q)) = t\,g(t,q)$

with $g(0,q) = (Xf)(q)$

Then

$$f(\varphi^X_{-t}(q)) = f(q) - t\,g(t,q)$$



[Y does not act on the variable t ..]

$$Y(f\circ\varphi^X_{-t})(q) = (Yf)(q) - t(Yg)(t,q)$$

Now, setting $q = \varphi^X_t(p)$

$\longrightarrow$

we have $\qquad\qquad q = \varphi^X_t(p)$

$$(\diamond) = \frac{-t(Yg)(t,\varphi^X_t(p)) + (Yf)(\varphi^X_t(p)) - (Yf)(p)}{t}$$

before taking lim $t \to 0$

$$= -(Yg)(t,\varphi^X_t(p)) + \frac{(Yf)(\varphi^X_t(p)) - (Yf)(p)}{t}$$

$g(t,\varphi^X_t(p)) \xrightarrow{t\to 0} (Xf)(p) \qquad t \to 0 \downarrow \qquad\qquad \mathcal{L}_X F = X(F)$

$$-(Y\circ X)f(p) + (XY)(f)(p)$$

$$= (XY - YX)(f)(p) \equiv [X,Y](f)(p)$$

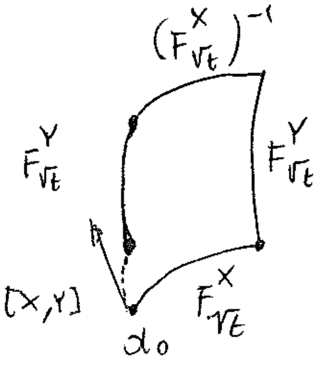
$\square$

— . — . — .

Y is said to be <u>invariant</u> under the flow of X if $\mathcal{L}_X Y = 0$. This is equivalent to $[X,Y] = 0$ and hence to $\mathcal{L}_Y X = 0$. Two such flows are said to commute: one has, in fact,

$$F^X_t \cdot F^Y_s = F^Y_s \cdot F^X_t \qquad (\forall\, s,t) \quad \text{if and only if}$$

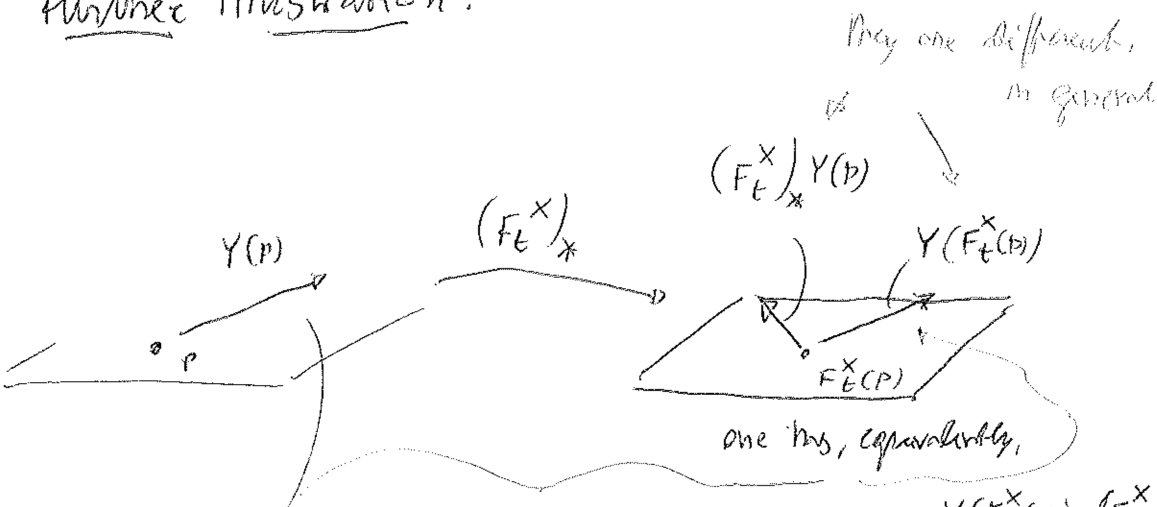
$[X,Y] = 0$. <u>The</u> <u>Lie</u> <u>bracket</u> <u>measures</u> <u>the</u> "<u>degree</u>" <u>of</u> <u>non</u> <u>commutativity</u> of <u>two</u> <u>flows</u>



one gets a curve whose tangent is, at $t=0$, $[X,Y](x_0)$

we shall deal with explicit examples in the sequel.

Further illustration:



one has, equivalently,

$$\mathcal{L}_X Y = \lim_{t \to 0} \frac{Y(F^X_t(p)) - (F^X_t)_* Y(p)}{t}$$

They cannot be compared directly, since they live in different spaces:

⚠ $Y(F^X_t(p)) - Y(p)$ does not make sense

$\therefore$ Y is invariant under the flow of X

precisely when $(F^X_t)_* Y(p) = Y(F^X_t(p))$ $\forall t, \forall p$,

(equiv: $(F^X_t)^{-1}_* Y(F^X_t(p)) = Y(p)$)