University of Verona
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Second Partial Test in Optimization

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Exercise 1. Consider the following control system in $\mathbb{R}^2$:

$$\begin{cases} \dot{x}_1(t) = x_1(t) - 3x_2(t); \\ \dot{x}_2(t) = u(t) + x_1(t) - 4x_2(t). \end{cases}$$

where $u(\cdot) \in \mathscr{U} := \{u : \mathbb{R} \to U := [-1, 1] \text{ measurable}\}$, and the problem $\max\limits_{u \in \mathscr{U}} x_1(T)$.

(1) Write the adjoint system with terminal conditions and solve it explicitely;
(2) Apply Pontryagin's Maximum Principle to determine the candidates and establish whether they are optimal in the case $T = \sqrt{2}\pi$.

Exercise 2. For $i = 1, 2, 3$, consider the following problem: minimize the functional

$$\int_0^1 \left(t^2 x'(t) + 2x'(t)^2 + x(t)^2 x'(t) + 2x(t)^2 + 3x(t)\right) dt, \qquad x(\cdot) \in \mathscr{C}_i,$$

where

$$\mathscr{C}_1 := C^2(\mathbb{R}), \quad \mathscr{C}_2 := \left\{x(\cdot) \in C^2(\mathbb{R}) : x(0) = 0, \, x(1) = 1\right\}, \quad \mathscr{C}_3 := \left\{x(\cdot) \in C^2(\mathbb{R}) : \int_0^1 x(t) \, dt = 1\right\}.$$

In each case establish wheater the infimum is attained or not, and in the positive case find explicitly the point of minimum.

Exercise 3.

(1) State and prove Ekeland's Variational Principle.
(2) Formulate and discuss the brachistochrone problem.
(3) For any $n \in \mathbb{N}$, let $p_n : \mathrm{Mat}_{n\times n}(\mathbb{R}) \to \mathrm{Mat}_{n\times n}(\mathbb{R})$ defined by $p_n(A) = A^n$ for all matrices $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$, where A^n denotes the usual product of matrices between n copies of A. Compute the Frechét differential of p_n w.r.t. a chosen norm on $\mathrm{Mat}_{n\times n}(\mathbb{R})$. Does the differential depends on the norm chosen on $\mathrm{Mat}_{n\times n}(\mathbb{R})$?
(4) State Pontryagin's Maximum Principle for a Mayer control problem.
(5) Provide a sufficient condition granting the convexity of the reachable set of a control system from a point.