Mathematical Logic
Problems 1

1. Using results from the lectures or otherwise, prove for every relation symbol R that
   a) $\mathrm{DNA}_R \vdash \mathrm{EFQ}_R$
   b) $\mathrm{DNA} \vdash \mathrm{TND}_R$
   c) $\mathrm{TND}_R, \mathrm{EFQ}_R \vdash \mathrm{DNA}_R$

   Conclude that $\Gamma \vdash_c A \iff \Gamma \cup \mathrm{TND} \vdash_i A$ for every formula A.

2. Prove for all formulas A, B that $\vdash_c ((A \to B) \to A) \to A$ (<u>Peirce formula</u>); For <u>atomic</u> formulas A, B find the Gödel-Gentzen translation of the Peirce formula and deduce this translation with minimal logic.

3. The essence of <u>Russell's Paradox</u> is that there cannot be any set S for which $S \in S \iff S \notin S$. Prove more generally that $(A \to B) \leftrightarrow A \vdash B$ for arbitrary formulas A, B and tell why this is a generalisation of Russell's Paradox.

(Recall that $\vdash$ stands for $\vdash_m$, for deducibility with minimal logic, whereas $\vdash_i$ and $\vdash_c$ stand for deducibility with intuitionistic and classical logic, respectively.)

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4. A formula of <u>propositional logic</u> is formed as a formula of predicate logic, but
   - <u>without</u> variables, function symbols, quantifiers;
   - <u>only from</u> nullary relation symbols, i.e. proposition symbols, as atomic formulas.

   a) Give an inductive definition of "A is a formula of propositional logic". Does propositional logic have terms?

   b) Prove for every formula A of propositional logic and for every set Γ of such formulas:

      i) $\vdash_i A^g \Leftrightarrow \neg\neg A$ (A^g: Gödel-Gentzen translation)

      ii) $\Gamma \vdash_c A \Leftrightarrow \neg\neg\Gamma \vdash_i \neg\neg A$ (<u>Glivenko's Theorem</u>)

5. Let $\mathcal{L} = (e, \_^{-1}, \circ, =)$ be the languge of <u>group theory</u>. Prove or disprove

   a) $\models \forall x \exists y\; x \circ y = e$

   b) $\models \forall x \forall y\; x \circ y = y \circ x$

   c) $\models \exists x.\, e = x \to \forall y\; e = y$

(Note that no axioms are assumed yet; whence the structures under consideration need not be groups – they just have to come with interpretations of the symbols.)

6. Let $\mathcal{L} = \{0, 1, +, \times, =\}$ be the language of <u>semirings</u>, that is: $0, 1$ are constants; $+, \times$ are binary function symbols; and $=$ is a binary relation symbol. Let the $\mathcal{L}$-structures $\mathcal{M}_1$, $\mathcal{M}_2$ be defined by $|\mathcal{M}_1| = \mathbb{Z}$ and $|\mathcal{M}_2| = \mathbb{Z}^{2\times 2}$ (2×2 matrices with integer entries), and by the usual interpretations of the symbols of $\mathcal{L}$.

Find $\mathcal{L}$-formulas A_1, A_2 such that

a) $\mathcal{M}_1 \models A_1$ and $\mathcal{M}_2 \not\models A_2$,

b) $\mathcal{M}_2 \not\models A_2$ and $\mathcal{M}_2 \models A_2$.

Prove what you are claiming(*)

7. Prove some (all?) of the statements that have been left as exercise at the lectures.

(*) (While it is irrelevant what the axioms of a semiring are, the properties of the interpretations of $0, 1, +, \times, =$ in $\mathcal{M}_1$ and $\mathcal{M}_2$ do matter, and of course are the usual, well-known ones.)