Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture XIV

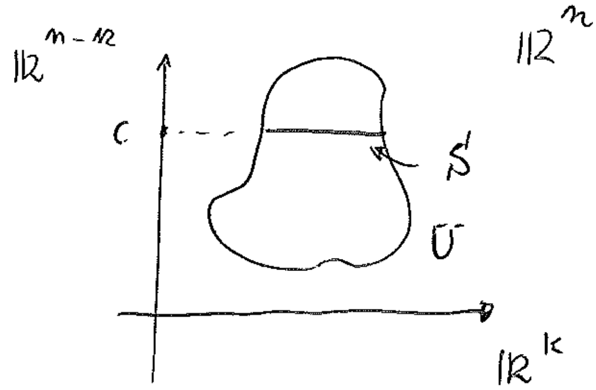
★ <u>$\mathbb{R}$-slices</u>

The sets of the form

$$S = \{(x^1 \ldots x^k, x^{k+1}, \ldots x^n) \in U \;/\; x^{k+1} = c^{k+1}, \ldots x^n = c^n\}$$

U open set in $\mathbb{R}^n$; $c^{k+1}, \ldots, c^n$ fixed

are called $\mathbb{R}$-slices of U



★ <u>Extension</u> of the <u>fundamental</u> <u>theorems</u> of <u>analysis</u> in <u>several</u> <u>variables</u> to <u>manifolds</u>.

★ The <u>Inverse</u> <u>function</u> <u>theorem</u>

$$\psi : X \to Y \qquad \psi \text{ smooth}$$
$$\dim X = \dim Y = n$$

Let $\psi_*|_{x_0} : T_{x_0}X \to T_{y_0}Y$, $y_0 = \psi(x_0)$, be an isomorphism.

Then $\exists\ \mathcal{U}_0 \ni x_0$ such that
neighbourhood of x_0

$$\Psi|_{U_0} : \mathcal{U}_0 \longrightarrow \Psi(\mathcal{U}_0)$$

is a diffeomorphism

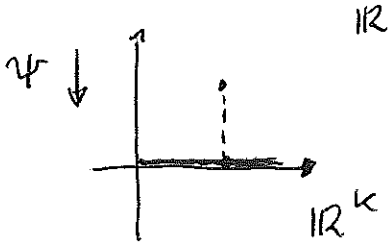
In coordinates, the proof reduces to the standard one, via the contraction lemma (Banach-Caccioppoli theorem)

$\ast$ The rank theorem

Let $\Psi : M \longrightarrow N$ $\quad (\dim M = m, \ \dim N = n)$
be smooth, with constant rank k $\quad k \leq m$. Then
$\forall\, p \in M$, there exist coordinates $(x^1 \dots x^m)$ centred at p and $(y^1, \dots, y^n)$ centred at $\Psi(p)$ such that

$$\Psi(x^1 \dots x^k, x^{k+1} \dots x^m) = (x^1 \dots x^k, 0, \dots 0)$$

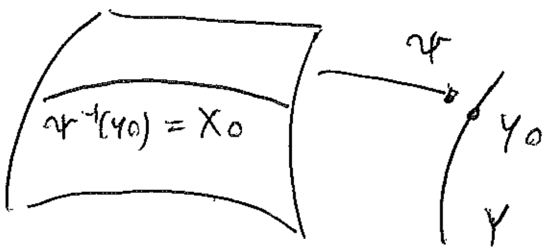
a slice in $\mathbb{R}^n$ $\qquad k \leq \min(m,n)$



$\ast$ The implicit function theorem

Let $\Psi : X \longrightarrow Y$ smooth, $\dim X = n >$
$\dim Y = m$

Let $y_0 \in Y$ and $X_0 = \Psi^{-1}(y_0) = \{ x \in X \,/\, \Psi(x) = y_0 \}$



Assume that
$\Psi_*|_x : T_x X \longrightarrow T_{\Psi(x)} Y$
is surjective $\forall x \in X_0$.

That is, Ψ is submersive $(\forall x \in X_0)$.

Then X_0 is a manifold (equipped with the relative topology inherited from X), and $X_0 \hookrightarrow X$ (inclusion) is smooth. Moreover $\dim X_0 = \dim X - \dim Y$
X_0: level manifold (of Ψ) $= n - m$.
varietà di livello

<u>Proof</u>. (Sketch) Let $V \ni y_0$ a coordinate neighbourhood (of y_0) with local coordinates $(y^1 .. y^m)$. Let $x_0 \in X_0$ and $U \ni x_0$ (coord. neighbourhood), with local coordinates $(x^1 .. x^n)$ centred at x_0 $(x^i(x_0) = 0, i = 1..n)$. Since $\Psi_*|_{x_0}$ is surjective, the Jacobian matrix

$$\left(\frac{\partial}{\partial x^j} (y^i \circ \Psi)\Big|_{x_0} \right)_{\substack{i=1...m \\ j=1..n}}$$

has rank m, so, up to a coordinate relabelling one can assume it to be of the form

$$\left(* \;\middle|\; \underset{m \times m,\ \text{non singular}}{J} \right)$$

Define $\tilde{\Psi} : U \to \mathbb{R}^{n-m} \times \bar{V}$ in the following manner:

$$\tilde{\Psi}(x) = (x^1(x) ... x^{n-m}(x), \Psi(x))$$

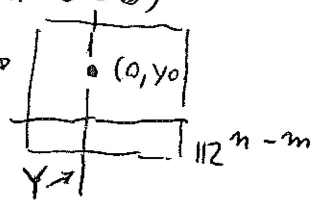
$$\Rightarrow \quad \tilde{\Psi}_*|_{x_0} \rightsquigarrow \begin{pmatrix} I_{n-m} & 0 \\ * & J \end{pmatrix}$$

examples on pages 6,7

which is an isomorphism. Therefore, by virtue of the inverse function theorem, $\exists\ U_0 \ni x_0$ such that $\tilde{\psi}|_{U_0}$ is injective, $\tilde{\psi}(U_0)$ is open in $\mathbb{R}^{n-m} \times V$ and $\tilde{\psi}^{-1} : \tilde{\psi}(U_0) \to U_0$ is smooth.

Without loss of generality (w.l.o.g.), $\tilde{\psi}(U_0)$ can be taken of the form $\underset{\ni\, 0}{W_0} \times \underset{\ni\, y_0}{V_0} \rightsquigarrow$

open hyperparallelopodes $\to$



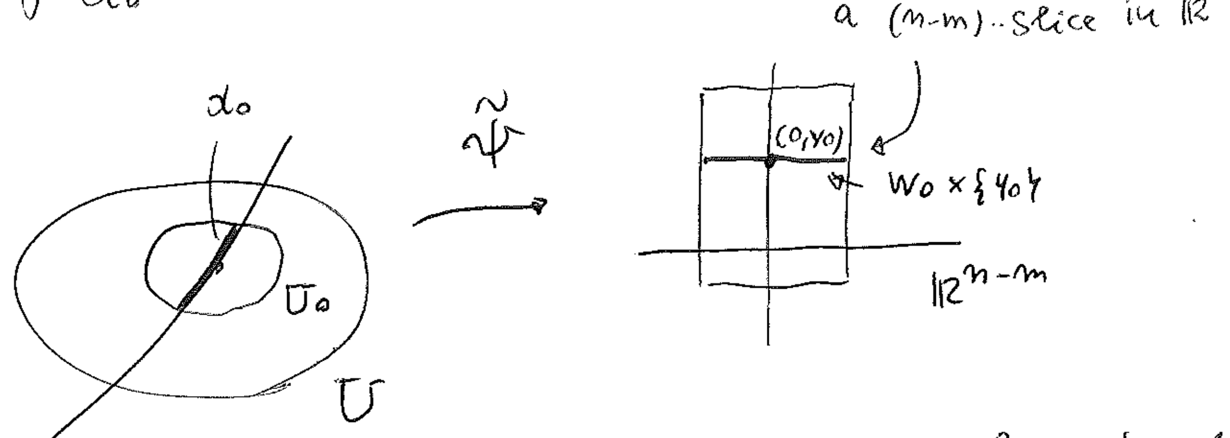
(open sets of this type yield a basis for the topology of $\mathbb{R}^{n-m} \times V$.

Now $\tilde{\psi}^{-1}(W_0 \times \{y_0\}) = X_0 \cap U_0$.

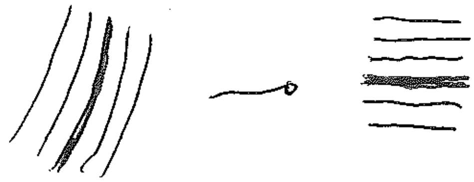
Since $\tilde{\psi}|_{U_0}$ is a homeomorphism, the map

$$\tilde{\psi}\Big|_{X_0 \cap U_0} : X_0 \cap U_0 \xrightarrow{\text{homeom.}} W_0 \times \{y_0\} \underset{W_0\ \subset\ \mathbb{R}^{n-m}}{\cong}$$

is a coordinate system in a neighbourhood of x_0



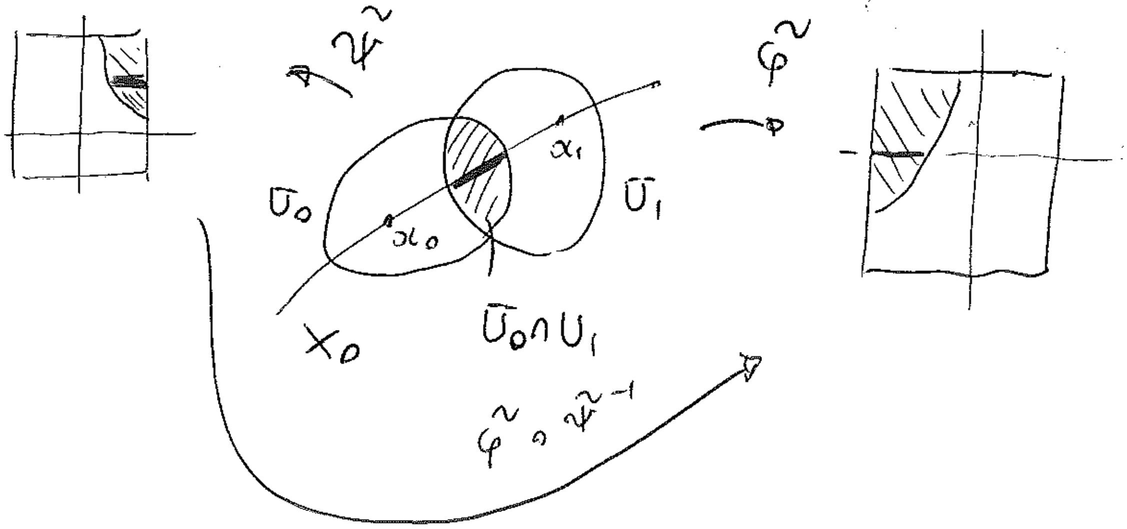
This is called a $(n-m)$-slice chart



level sets of ψ

One is left with checking the behaviour on non empty intersections

$$\tilde\psi : \bar U_0 \to W_0 \times V_0$$
$$\tilde\varphi : \bar U_1 \to W_1 \times \bar V_1$$



$\tilde\varphi \circ \tilde\psi^{-1}\big|_{\tilde\psi(\bar U_0 \cap \bar U_1)}$ is $\mathscr{C}^\infty$, and so is

$$\tilde\varphi \circ \psi^{-1}\big|_{\tilde\psi(X_0 \cap U_0 \cap \bar U_1)} : \tilde\psi(X_0 \cap \bar U_0 \cap \bar U_1) \to \tilde\varphi(X_0 \cap \bar U_0 \cap \bar U_1)$$

$$\tilde\psi(X_0 \cap U_0 \cap \bar U_1) = (\tilde\psi(\bar U_0 \cap \bar U_1) \cap \mathbb{R}^{n-m}) \times \{y_0\}$$

(the same being true for $\tilde\psi \circ \varphi^{-1}$),

and this yields the desired conclusion.

★ Dini's theory revisited
via the inverse function theorem
explicit examples

★ curves in $\mathbb{R}^2$ $\qquad f: \mathbb{R}^2 \to \mathbb{R}$ $\qquad f(x,y) = 0$
(or more generally $f(x,y) = c$)

$P_0 : (x_0, y_0)$ $\qquad f(P_0) = 0$

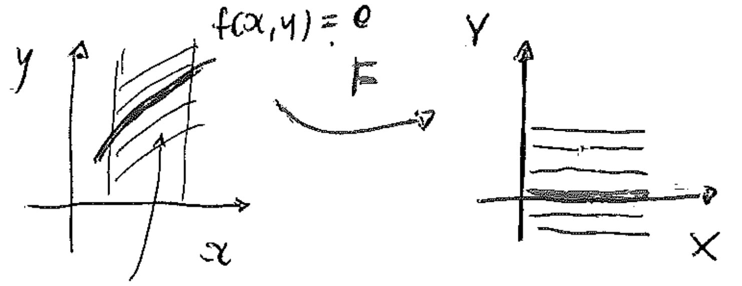
$\frac{\partial f}{\partial y}(P_0) \neq 0 \Rightarrow$ loc. $\exists !\ y(x)$, $y_0 = y(x_0)$
with $f(x, y(x)) \equiv 0 \quad \forall x$ (in a suitable interval)

Define
$$F : \underset{\mathbb{R}^2}{(x,y)} \longmapsto (\overset{X}{x}, \overset{Y}{f(x,y)}) \in \mathbb{R}^2$$

$$F: \begin{cases} X = x \\ Y = f(x,y) \end{cases} \qquad \begin{aligned} dX &= dx \\ dY &= f_x\, dx + f_y\, dy \end{aligned}$$

$$F_* : \begin{pmatrix} 1 & 0 \\ f_x & \boxed{f_y} \end{pmatrix} \qquad f^0_y \neq 0 \Rightarrow$$
F_*^0 isomorphism

$\Rightarrow$ locally F is a diffeom.



1-slice chart

level sets are "rectified"

★ **surfaces in $\mathbb{R}^3$**

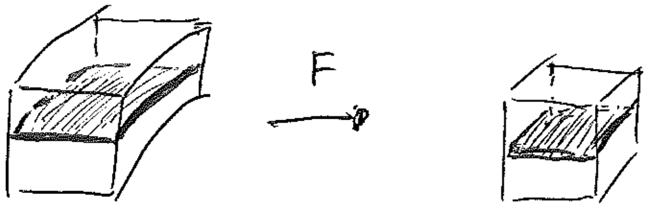
$$f(x_0, y_0, z_0) = 0 \quad (P_0)$$

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad f(x,y,z) = 0 \qquad f_z^0 \neq 0$$

$$F : \underset{\mathbb{R}^3 \ni}{(x,y,z)} \longmapsto (\underset{X}{x}, \underset{Y}{y}, \underset{Z}{f(x,y,z)}) \in \mathbb{R}^3$$

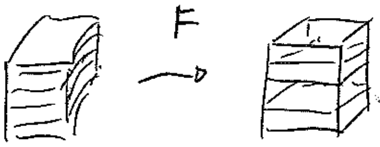
$$F_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{pmatrix} \qquad f_z^0 \neq 0 \Rightarrow F_*^0 \text{ isomorphism}$$

$\Rightarrow$ locally F is a diffeom.



2-slice in $\mathbb{R}^3$

2-slice chart



★ **curves in $\mathbb{R}^3$**

$$F : \mathbb{R}^3 \to \mathbb{R}^3 \qquad \begin{cases} f = 0 \\ g = 0 \end{cases}$$

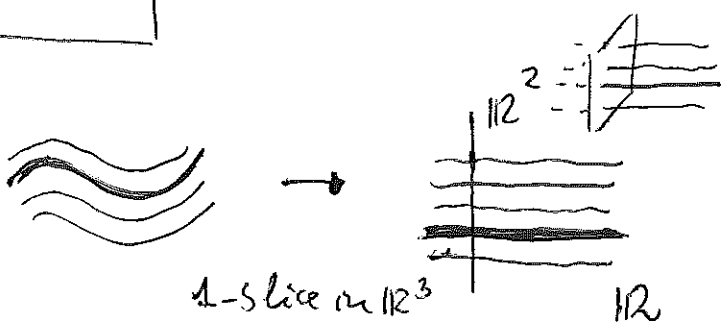
$$(x,y,z) \longmapsto (\underset{X}{x}, \underset{Y}{f(x,y,z)}, \underset{Z}{g(x,y,z)})$$

$$\boxed{\begin{array}{c} \mathbb{R}^3 \to \mathbb{R}^2 \\ (x,y,z) \xmapsto{(f,g)} (0,0) \end{array}}$$

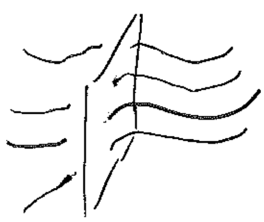
$$F_* = \begin{pmatrix} 1 & 0 & 0 \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix} \qquad \frac{\partial(f,g)}{\partial(y,z)}(P_0) \neq 0$$

F_*^0 isom.

$\Rightarrow F$ local diffeom.



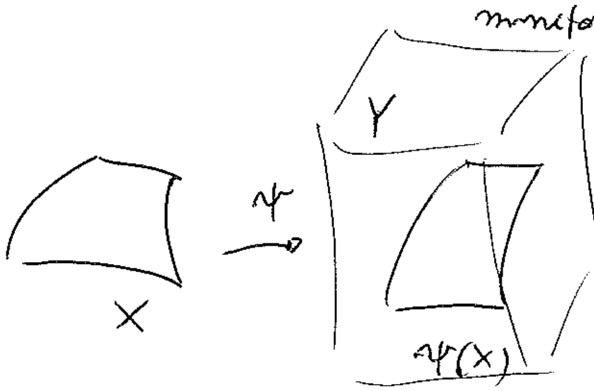
1-slice in $\mathbb{R}^3$



sottovarietà immersa

Def. (i) An immersed submanifold of a smooth manifold Y is a pair (X, ψ), $\psi : X \to Y$, smooth, with X a smooth manifold, with ψ and ψ_* injective

(ψ: injective immersion)



[with an abuse of language one identifies then X with its image $\psi(X) \subset Y$]

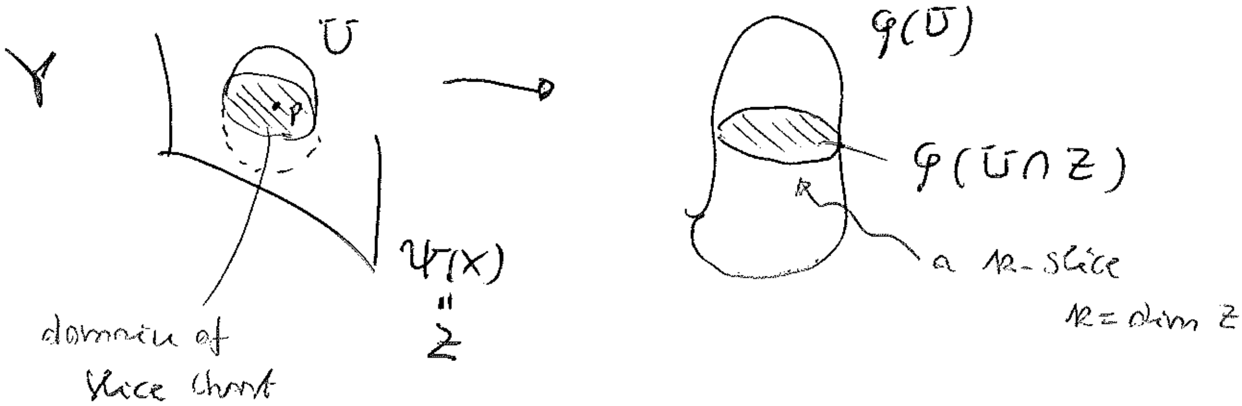
(ii) One has an embedding if, in addition to (i), inclusione

$\psi : X \to \psi(X)$ is a homeomorphism

(with $\psi(X)$ equipped with the relative topology induced from Y)

☆☆☆ Let us check that, if ψ is an embedding, then $\psi(X)$ is a submanifold of Y in the following sense:

Every point of $\psi(X)$ admits a coordinate neighbourhood $U \subset Y$ such that $\psi(X) \cap U$ is the domain of a slice-chart



Remark. The level submanifolds previously discussed are indeed submanifolds of X in the above sense.

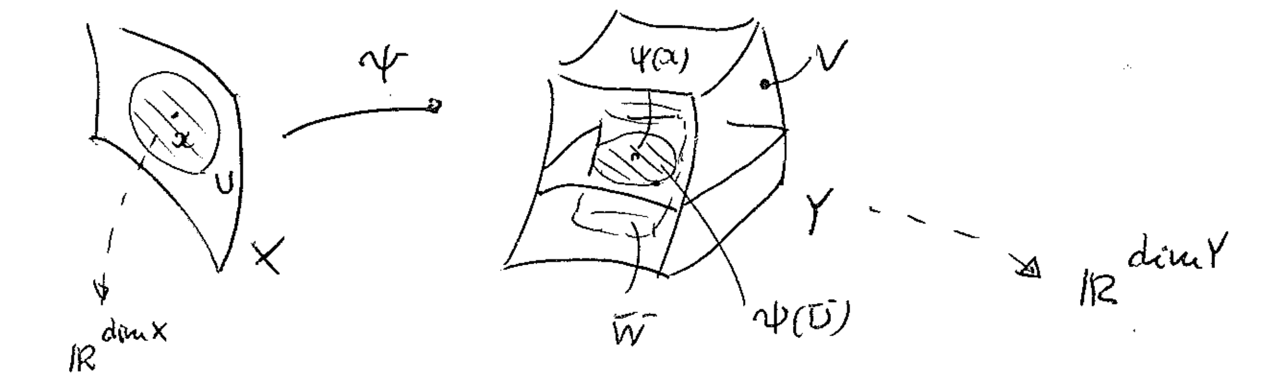
Proof. Let $x \in X$. Since an embedding has constant rank, by the rank theorem one can find local coordinate systems centred at x and $\psi(x)$, respectively, with

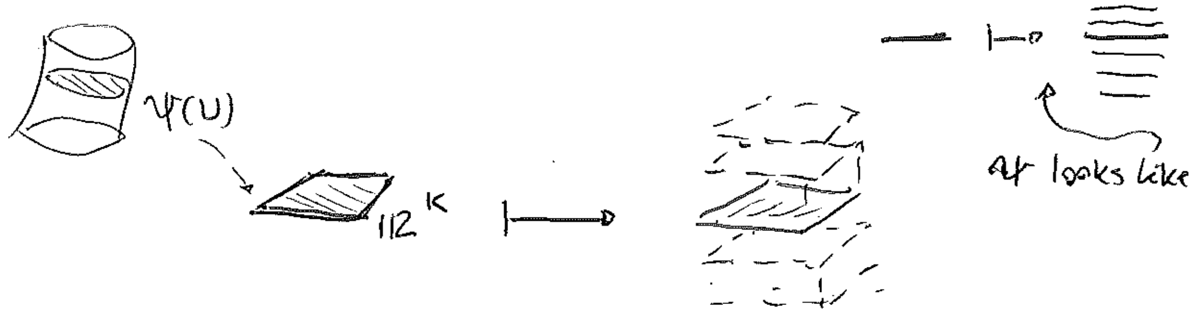
$$\underbrace{\psi : (x^1 \ldots x^k)}_{U} \longmapsto \underbrace{(x^1 \ldots x^k, 0 \ldots 0)}_{V}$$

abuse of notation $\rightsquigarrow$ $k = \dim X$,   number of zeros $= \dim Y - \dim X$

upon possible restriction of $\bar V$, $\psi(\bar U)$ becomes a slice in Y. Now $\psi(\bar U)$ is open in $\psi(X)$ (ψ is a homeomorphism onto $\psi(X)$) $\Rightarrow \exists\, W \subset Y$ such that $\psi(\bar U) = \bar W \cap \psi(X)$ ⚠ this is the crucial point

Let $\hat V = V \cap \bar W$. One produces a slice chart such that $\hat V \cap \psi(X) = \hat V \cap \psi(\bar U)$ is a slice in $\tilde V$





Notice that if X is compact, ψ is automatically a homeomorphism onto $\psi(X)$ (since $\psi(X)$ is Hausdorff and ψ is injective)

In any case, in view of Dini's theory,

ψ injective + ψ_* injective $\Rightarrow$ ψ homeomorphism (locally)

i.e. an injective immersion is locally an embedding

★ Summary. F_* (differential, or push-forward)

slogan " F_* behaves locally like F "

★ crucial tools

| inverse function theorem |

⇓

- rank theorem
- implicit function theorem

important case: surjective submersions

$F : M \to N$

rank of F at $p \in M$ $\equiv$ rank of $F_* : T_pM \to T_{F(p)}N$

if this does not vary with p, we say F of constant rank

F: submersion : F_* surjective $\forall p \in M$ rank $r(F) = \dim N$

F: immersion : F_* injective $r(F) = \dim M$

F: embedding : F immersion + F homeomorphism onto $F(M)$

"topological embedding"

equivalently :

F injective

F_* injective

$F : M \to F(M)$ homeomorphism

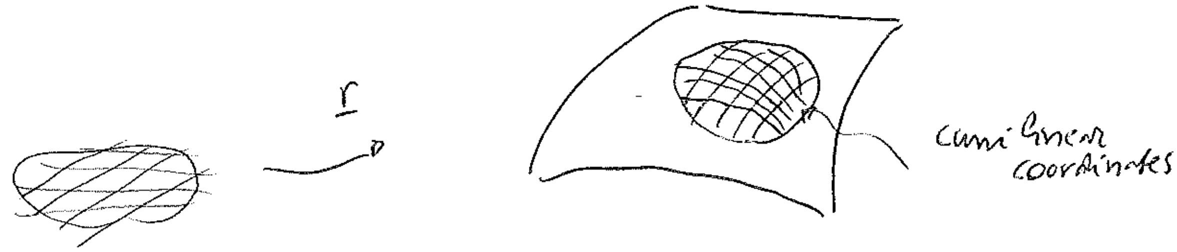
equipped with the relative topology inherited from N

★ Notice that we obviously recover the elementary treatment of (parametric) surfaces. Remember:

1. $\underline{r} = \underline{r}(u,v)$ $\quad \mathcal{C}^{\infty}$ $\quad (u,v) \in \mathcal{U} \subset \mathbb{R}^2$ ($\mathcal{U}$ disc)

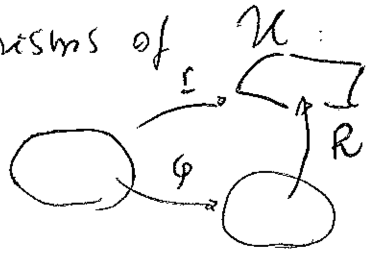
2. $\underline{r}$ injective

3. $\underline{r}_u \times \underline{r}_v \neq \underline{0}$



||| $\underline{r} = \underline{r}(u,v)$ fulfilling 1 – 3 is in fact an injective immersion (and locally an embedding)

What was not insisted upon in the Geometry course was invariance of conditions 1 – 3 under diffeomorphisms of $\mathcal{U}$: $\varphi : \begin{cases} u' = u'(u,v) \\ v' = v'(u,v) \end{cases}$



$$\underline{R} = \underline{r} \circ \varphi^{-1}$$

$$\underline{R}(u',v') = \underline{r}(u(u',v'), v(u',v'))$$

1 & 2 are clear for $\underline{R}$, and 3 is simple as well:

$$\underline{R}_{u'} \times \underline{R}_{v'} = \underbrace{\left(\frac{\partial u}{\partial u'}\frac{\partial v}{\partial v'} - \frac{\partial u}{\partial v'}\frac{\partial v}{\partial u'} \right)}_{\frac{\partial(u,v)}{\partial(u',v')} \neq 0} \underline{r}_u \times \underline{r}_v$$

$$\boxed{\begin{aligned} \frac{\partial \underline{R}}{\partial u'} &= \frac{\partial \underline{r}}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \underline{r}}{\partial v}\frac{\partial v}{\partial u'} \\ \frac{\partial \underline{R}}{\partial v'} &= \frac{\partial \underline{r}}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial \underline{r}}{\partial v}\frac{\partial v}{\partial v'} \end{aligned}}$$

$$\frac{\partial(u,v)}{\partial(u',v')} = \begin{vmatrix} u_{u'} & u_{v'} \\ v_{u'} & v_{v'} \end{vmatrix}$$

* Examples

1. $$F : \mathbb{R} \to \mathbb{R}$$
$$x \longmapsto e^x \qquad \mathrm{Im}(F) = \mathbb{R}^+$$

is a submersion which is not surjective

$$F_*|_x = e^x \qquad (de^x = e^x dx)$$

$$e^x : T_x\mathbb{R} \to T_{e^x}\mathbb{R}, \qquad T_x\mathbb{R} \cong \mathbb{R}, \quad T_{e^x}\mathbb{R} \cong \mathbb{R}$$

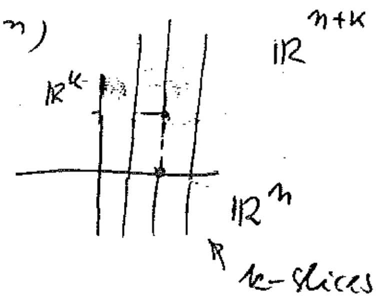
$$h \longmapsto e^x h$$ is surjective $\forall x \in \mathbb{R}$
(actually it is an isomorphism)

1'. $$F : \mathbb{R} \to \mathbb{R}^+$$
$$x \longmapsto e^x$$

is a surjective submersion

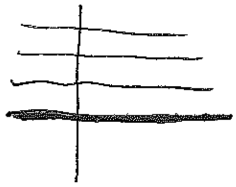
2. $$\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$$
$$(x^1 .. x^n, x^{n+1} .. x^{n+k}) \longmapsto (x^1 .. x^n)$$

is a surjective submersion



3. $$i : \mathbb{R}^n \to \mathbb{R}^{n+k}$$

$$(x_1 ... x_n) \longmapsto (x_1 ... x_n, 0, 0 .. 0)$$ embedding
or $c_1 ... c_k$



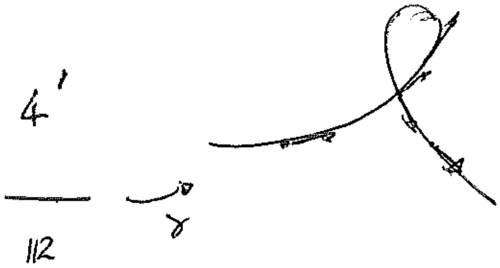
4. $\gamma : \underset{\text{open interval}}{I} \longrightarrow M$

γ immersion : $\gamma'(t) \neq 0 \quad \forall t \in I$

γ embedding : γ injective , $\gamma'(t) \neq 0 \ \forall t$, and

$\gamma : I \to \gamma(I)$ homeomorphism

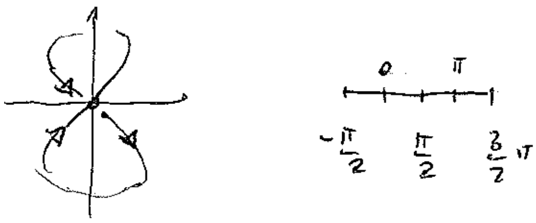
4'



This is an immersion (we require $\gamma'(t) \neq 0 \ \forall t \in I$)
but it is not an embedding
(γ is not injective, so it cannot be a homeomorphism)

4''. $\gamma : \overset{I}{\left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)} \longmapsto \mathbb{R}^2$

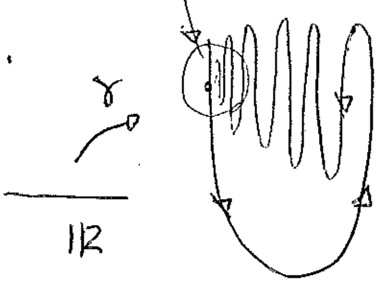
$\gamma(t) = (\sin 2t, \cos t) \qquad x^2 = 4y^2(1-y^2)$



I is not compact but $\gamma(I)$ is compact
($\Rightarrow$ γ cannot be a homeomorphism, since compactness is a topological property)

points like this one do not admit an arcwise connected neighbourhood

4'''.



This is not an embedding :
$\mathbb{R}$ is locally arcwise connected, but $\gamma(\mathbb{R})$ is <u>not</u> .

again a topological property

5. This is a very important example!

$$\gamma : \mathbb{R} \longrightarrow \mathbb{T}^2 = S^1 \times S^1$$
$$t \longmapsto \gamma(t) = \left(e^{2\pi i t}, e^{2\pi i c t}\right)$$
$$c \in \mathbb{R} \setminus \mathbb{Q}$$

$\mathrm{Im}\,\gamma$ is dense in $S^1 \times S^1 = \mathbb{T}^2$

$\mathbb{Z} \subset \mathbb{R}$ is a discrete set (i.e., without limit points), whilst $\gamma(\mathbb{Z}) \subset \mathbb{T}^2$ is not, whence γ cannot be a homeomorphism onto $\gamma(\mathbb{R})$.

Also, one observes that $\mathrm{Im}(\gamma)$ is not locally arcwise connected.

--- According to Whitney's theorem, every f.d. manifold can be embedded in a suitable $\mathbb{R}^N$. However, this property is more effective in theoretical matters than in practice.

J. Nash extended Whitney's results to isometric embeddings of Riemannian manifolds in $\mathbb{R}^N$ (so that the metric is inherited from $\mathbb{R}^N$), a major "tour-de-force"!

However, we close our discussion at this point.