Convolution

$f_1[k] \longleftrightarrow F_1[z]$

$f_2[k] \longleftrightarrow F_2[z]$

1. $f_1 * f_2 \leftrightarrow F_1[z] F_2[z]$ Time convolution

2. $f_1(k) f_2(k) \leftrightarrow \frac{1}{2\pi j} \oint F_1(u) F_2\left[\frac{z}{u}\right] u^{-1} du$

<u>LTID system response</u>

$y(k) = f[k] * h[k] \Rightarrow Y[z] = F[z] H[z]$

1. $f(k) u[k] \longleftrightarrow F[z]$

$\gamma^k f[k] u[k] \longrightarrow F\left[\frac{z}{\gamma}\right]$

Proof: $\mathcal{Z}\{\gamma^k f[k] u[k]\} = \sum_0^{+\infty} \gamma^k f(k) z^{-k} = \sum_0^{+\infty} f(k)\left(\frac{z}{\gamma}\right)^{-k} = F\left[\frac{z}{\gamma}\right]$ CQD

2. $k f[k] u[k] \leftrightarrow -z \frac{d}{dz} F[z]$

Proof:

$$-z \frac{d}{dz} F(z) = -\frac{zd}{dz} \sum_0^{+\infty} f(k) z^{-k} = -z \sum_0^{+\infty} -k f(k) z^{-k-1} =$$

$$= \sum_0^{+\infty} + k f(z) z^{-k} = \mathcal{Z}\{k f[k] u(k)\} \text{ CQD}$$

3. For a causal $f(k)$:

$$f[0] = \lim_{z \to \infty} F[z] \qquad \left(F[z] = \sum_0^{+\infty} f(k) z^{-k}\right)$$

$$\left(\lim_{N \to \infty} f(N) = \lim_{z \to 1} (z-1) F(z) \right)$$

Linear differential equations

Z-Transform converts linear difference equations into algebraic equations.
Key point: Time shifting
Ex. 11.5

In advance form we would need initial cond. of the type $y[0], y[1] \ldots$

$$y[k+2]-5y[k+1]+6y[k]=3f[k+1]+5f[k]$$

$$y[-1]=\frac{11}{6}$$

$$y[-2]=\frac{37}{36}$$

$\Rightarrow$ we must convert it into the delay form

$k \to k-2$

$$f[k]=2^{-k}u[k]$$

$$y[k]-5y[k-1]+6y[k-2]=3f[k-1]+5f[k-2] \qquad (1)$$

The use of unilateral transform implies that we are considering the situation for $k \geq 0$ and every signal has to be considered from $k=0$.

Note: $y[k-j] \Leftrightarrow y[k-j]u[k]$

since the use of the unilateral transform implies $k \geq 0$ and every signal must be counted from $k=0$.

Then:

$$y[k]u[k] \leftrightarrow Y[z]$$

$$y[k-1]u[k] \leftrightarrow \frac{1}{z}Y[z]+y[-1]=\frac{1}{z}Y[z]+\frac{11}{6}$$

$$y[k-2]u[k]=\frac{1}{z^2}Y[z]+\frac{1}{z}y[-1]+y[-2]=\frac{1}{z^2}Y[z]+\frac{11}{6z}+\frac{37}{36}$$

and:

$$f[k]=(2)^{-k}u[k]=(0,5)^k u[k] \leftrightarrow F(z)=\frac{z}{z-0,5}$$

$$f[k-1]u[k]=\frac{1}{z}F[z]+f[-1]=\frac{1}{z}\frac{z}{z-0,5}+0=\frac{1}{z-0,5} \qquad \text{(causal)}$$

$$f[k-2]u[k]=\frac{1}{z^2}F[z]+\frac{1}{z}f[-1]+f[-2]=\frac{1}{z(z-0,5)}$$

Replacing in (1) after taking the Z-transf.:

$$Y[z]-5\left[\frac{1}{z}Y(z)+\frac{11}{6}\right]+6\left[\frac{1}{z^2}Y(z)+\frac{11}{6z}+\frac{37}{36}\right]=\frac{3}{z-0,5}+\frac{5}{z(z-0,5)}$$

$$\left(1-\frac{5}{z}+\frac{6}{z^2}\right)Y(z)-\left(3-\frac{11}{z}\right)=\frac{3}{z-0,5}+\frac{5}{z(z-0,5)}$$

After some manipulations:

$$Y(z) = \frac{25}{16}\left(\frac{z}{z-0.5}\right) - \frac{7}{3}\left(\frac{z}{z-2}\right) + \frac{18}{5}\left(\frac{z}{z-3}\right)$$

$$\hookrightarrow \; y[k] = \left[\frac{25}{16}(0.5)^k - \frac{7}{3}(2)^k + \frac{18}{5}(3)^k\right] u[k]$$

Zero input and zero-state responses can be easily separated.

Ex:

$$\left(1 - \frac{5}{z} + \frac{6}{z^2}\right) Y(z) - \underbrace{\left(3 - \frac{11}{z}\right)}_{\text{initial conditions}} = \underbrace{\frac{3}{z-0.5} + \frac{5}{z(z-0.5)}}_{\text{resulting from the input.}}$$

$$\frac{z^2 - 5z + 6}{z^2} Y(z) = \frac{3z - 11}{z} + \frac{3z + 5}{z(z-0.5)}$$

$$Y(z) = \underbrace{\frac{3z - 11}{z^2 - 5z + 6}}_{\text{zero input}} + \underbrace{\frac{z(3z+5)}{z - 0.5}}_{\text{zero-state}}$$

Sometimes, auxiliary cond. $y[0], y[1], \ldots, y[n-1]$ are given instead of initial conditions $y[0], y[-1] \ldots, y[-n]$.
In this case the advance form is more convenient.
(see ex. 11.9)

Transfer function: Zero state response

$$Q[E]\, y[k] = P[E] f[k] \qquad (2)$$

Zero-state response:

initial cond. $y[-1] = y[-2] = \cdots = y[-n] = 0 \quad \Rightarrow \quad y[k-m]\, u[k] \Leftrightarrow \frac{1}{z^m} Y[z]$

Causal input:

$$f[k] = 0 \quad \forall\, k < 0$$

$$f[k-m]\, u[k] \Leftrightarrow \frac{1}{z^m} F[z]$$

$$(2) \quad y[k] + a_{n-1}\, y[k-1] + \cdots + a_0\, y[k-n] = b_n f[k] + \cdots + b_0 f[k-n]$$

$$\Rightarrow \quad Y[z] + \frac{1}{z} a_{n-1} Y[z] + \cdots + a_0 \frac{1}{z^n} Y[z] = b_n F[z] + \cdots + \frac{1}{z^n} b_0 F[z]$$

$$\left(1+\frac{1}{z}a_{n-1}+\cdots+\frac{1}{z^n}a_0\right)y[z]=\left(b_n+\frac{1}{z}b_{n-1}+\cdots+\frac{1}{z^n}b_0\right)f[z]$$

$$\Rightarrow \underbrace{\left(z^n+a_{n-1}z^{n-1}+\cdots+a_0\right)}_{Q(z)}y[z]=\underbrace{\left(b_nz^n+b_{n-1}z^{n-1}+\cdots+b_0\right)}_{P(z)}F[z]$$

$$\Rightarrow \boxed{Y(z)=\frac{P(z)}{Q(z)}F(z)}$$

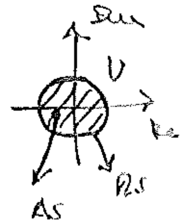
$$H(z)=\frac{P(z)}{Q(z)}=\frac{b_nz^n+b_{n-1}z^{n-1}+\cdots+b_1z+b_0}{z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0} \Rightarrow H[z]=\frac{Z[\text{zero-state response}]}{Z[\text{input}]}$$

$$Y[z]=H(z)F[z]$$

$$F[z] \longrightarrow \boxed{H(z)} \longrightarrow Y(z)$$

⇒ The poles of $H(z)$ are the characteristic roots of the system

⇒ stability analysis, based on the poles of the transfer function



Bilateral Z-Transform : the input signals are not constrained to be causal.

$$\begin{cases} F(z)=\sum_{k=-\infty}^{+\infty} f(k)\,z^{-k} \\ f(k)=\dfrac{1}{2\pi j}\oint F(z)\,z^{k-1}\,dz \end{cases}$$

Unilateral and bilateral transforms can lead to the same $F[z]$ for different $f(k)$ with different ROC ⇒ the specification of the ROC is needed to be able to invert the transform.

This ambiguity is removed if only the unilateral transform is considered.

Example:

Unilateral: $\gamma^k u[k] \leftrightarrow \dfrac{z}{z-\gamma}$ ROC: $|z| > |\gamma|$

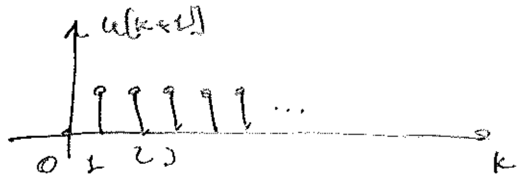
Bilateral: $-\gamma^k u[-(k+1)] \leftrightarrow \dfrac{z}{z-\gamma}$ ROC: $|z| < |\gamma|$

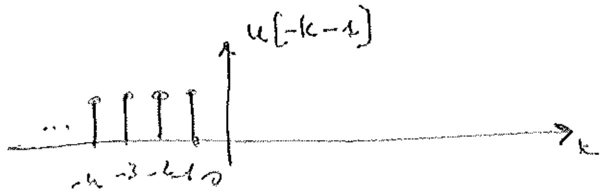
Proof:

$$\mathcal{Z}\{-\gamma^k u[-(k+1)]\} = \sum_{-\infty}^{-1} -\gamma^k z^{-k} = \sum_{-\infty}^{-1} \left(\frac{\gamma}{z}\right)^k = 1 - \sum_{0}^{+\infty} \left(\frac{z}{\gamma}\right)^k =$$

$$= 1 - \frac{1}{1-\frac{z}{\gamma}} = \frac{z}{z-\gamma} \quad \text{CQD}$$

$$\text{ROC: } \left|\frac{z}{\gamma}\right| < 1 \Rightarrow |z| < |\gamma|$$





* The ambiguity can be removed by restricting to causal signals. *

11.7.1 Analysis of LTID systems using the bilateral Z-Transform

Important for non causal signals (ex. images)

Zero-state response:

$$y[k] = \mathcal{Z}^{-1}\{F(z)H(z)\}$$

provided that $F(z)H(z)$ exists.

The ROC is the one where both $F(z)$ and $H(z)$ exist!

(and elec-11.13 and 11.14)

Take home messages

1. The Z-Transform is a generalization of the DTFT with the frequency variable $j\Omega$ generalized to $\sigma + j\Omega$;

2. Z-Transf. changes difference equations to algebraic equations;

3. The transfer function of an LTID system, $H(z)$, is equal to the ratio of the Z-Transf. of the input when all initial conditions are zero;

4. $Y[z] = H[z]\,X[z]$

5. $H[z] = \mathcal{Z}[h(k)]$

6. The system response to z^k is $H[z]\,z^k$;

7. The system stability depends on the position of the poles of $H(z)$ in the complex plan.