Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture I

★ Dual spaces

Let (V, K) be a vector space (over a field K; $K = \mathbb{R}$ or $\mathbb{C}$ throughout). The dual vector space V^* is, by definition

$$V^* = \{ f : V \to K \;/\; f \text{ linear} \}$$

[terminology: the elements in V^* are called linear functions / linear functionals / linear forms / (algebraic) 1-forms]

In more detail:

$$f(\underset{\in V}{\overset{K \ni}{\alpha} \cdot v} + \underset{\in V}{\overset{K \ni}{\beta} \cdot w}) := \underbrace{\overset{K \ni}{\alpha} \cdot f(v)}_{\in K} + \underbrace{\overset{K \ni}{\beta} f(w)}_{\in K}$$

operations in V operations in K

★ V^* is actually a vector space upon defining linear combinations in the following fashion:

this is a function, so one has to specify its action on elements in its domain

$$\underbrace{(\alpha \cdot f + \beta \cdot g)}_{\in V^*}(\underset{\in V}{v}) := \alpha \cdot f(v) + \beta\, g(v)$$

operations in K

operations in V^*, defined via

and checking vector space axioms.

☆ Let $\dim_K V = n < \infty$ (finite dimensional vector space)

Then $\dim_K V^* = n$ (hence $V \cong V^*$)
* isomorphic

Pf. Let $(e_1 \dots e_n)$ be a basis of V. Consider the dual forms $\{e^*_i\}_{i=1\dots n}$, defined via

$$e^*_i(e_j) = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

Kronecker's delta

* We want to show that $(e_1^* \dots e_n^*)$ is a basis for V^*, called the <u>dual basis</u> of $(e_1 \dots e_n)$

[recall that it is enough to define $f \in V^*$ on a basis of V, and "extend by linearity", since $f(v) = f(\sum_i \underset{\substack{\uparrow \\ \text{components of } v \\ \text{w. resp. to } (e_1 \dots e_n)}}{\alpha_i} e_i) = \sum \alpha_i f(e_i)$]. Observe that e_j^* picks up the j^{th} component of v = linear combination of the e_i's (components are uniquely defined) : $e_j^*(v) = \alpha_j$

One immediately finds:

$$f = \sum_{i=1}^{n} f(e_i) e_i^*$$

(that is: the e_i^*'s <u>generate</u> V^*)

Indeed, if $v = \sum_{i=1}^{n} \alpha_i e_i$, then, on the one hand,

$$f(v) = \sum_{i=1}^{n} \alpha_i f(e_i)$$

and, on the other hand,

$$\left(\sum_{i=1}^{n} f(e_i) e_i^*\right)(v) = \sum_{i,j=1}^{n} \alpha_j f(e_i) \underbrace{e_i^*(e_j)}_{= \delta_{ij}} = \sum_{i=1}^{n} \alpha_i f(e_i) .$$

Furthermore, the e_i^*'s are <u>linearly independent</u> :

If $\sum \beta_i e_i^* = 0$ ~ (the zero-functional)

then, $\forall v \in V$, $(\sum \beta_i e_i^*)(v) = 0$. Choosing $v = e_j$

yields $\quad 0 = \sum_i \beta_i e_i^*(e_j) = \sum_i \beta_i \delta_{ij} = \beta_j ,$

i.e. $\beta_j = 0 \quad \forall j = 1 \dots n$, whence the conclusion. $\square$

$\star$ Notice that $V \cong V^*$, but non-canonically (i.e. the established isomorphism is <u>basis-dependent</u>).

Define $V^{**} = (V^*)^* \equiv$ <u>bidual</u> of V

In finite dimensions, $V \cong V^{**}$ <u>canonically</u> (i.e. <u>independently</u> of the choice of a basis): this follows from setting, for any $v \in V$,

$$v^{**} \in V^{**} , \text{ defined via}$$

$$v^{**}(\underset{\underset{V^*}{\cap}}{f}) := f(v) \qquad (f \in V^*)$$

The map $\quad V \ni v \longmapsto v^{**} \in V^{**}$

is linear, injective, and $\dim V^{**} = n$, hence it is surjective as well (in view of the nullity + rank theorem), so it is an isomorphism.

Examples

1. $\mathbb{R}^n$ $(\mathbb{C}^n ..)$, $(e_1 \dots e_n)$ canonical basis

$$\mathbb{R}^n \ni x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad x_i \in \mathbb{R}$$

$$e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

dual basis $(e_1^* \dots e_n^*)$

$$e_i^* = (0, 0 \dots \underset{\uparrow \atop i}{1}, 0 \dots 0)$$

upon realizing

$$(\mathbb{R}^n)^* = \{ \overbrace{(a_1 \dots a_n)}^{a^T} \}_{a_i \in \mathbb{R}} \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

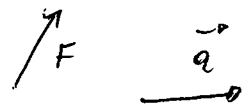
$$(a_1 \dots a_n) \mapsto f_a \qquad f_a(\underset{\mathbb{R}^n}{x}) = a^T x = \sum a_i x_i$$

$(a_1 \dots a_n)$ ↑ components of f with respect to the dual basis $(e_1^* \dots e_n^*)$

a^T x (matrix product)

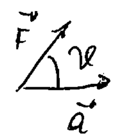
2. Within the geometric vector space :

$\vec{F}$: force $\vec{a}$: displacement



The <u>work</u> exerted by $\vec{F}$ along $\vec{a}$ is given by

$$\vec{F} \cdot \vec{a} = \|\vec{F}\| \cdot \|\vec{a}\| \cos \vartheta$$



(elementary scalar product)

$$l = l_{\vec{F}} \text{ , defined as } l_{\vec{F}}(\vec{a}) = \vec{F} \cdot \vec{a}$$

is a 1-form (<u>work</u> 1-form)

3. The differential of a function

Let $f : \mathcal{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, $f \in \mathcal{C}^1(\mathcal{U})$
open set — to fix ideas

Let $x_0 \in \mathcal{U}$, $x+h \in \mathcal{U}$. Then

$$f(x_0+h) - f(x_0) = \underbrace{df\big|_{x_0}}_{\text{linear part of the increment}} \cdot h + o(h) \qquad \left(\frac{\|o(h)\|}{\|h\|} \to 0 \right)$$

The linear operator

$$df\big|_{x_0} : \quad \begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R} \\ h &\longmapsto df\big|_{x_0} h \end{aligned}$$

(now defined for all $h \in \mathbb{R}^n$)

★ differential of f at x_0

is indeed a 1-form, and it is represented, concretely, by a $1 \times n$ - matrix

$$df\big|_{x_0} = \left(\frac{\partial f}{\partial x_1}(x_0), \dots \frac{\partial f}{\partial x_n}(x_0) \right) = \nabla f(x_0)^t$$

"gradient" (abuse of language ⚠) see also below

Aside
Fréchet differential:
$f : \mathcal{U} \subset V \longrightarrow W$ normed vector spaces

$$f(a+h) - f(a) = T_a h + o(h) \qquad \frac{\|o(h)\|_W}{\|h\|_V} \to 0 \text{ as } h \to 0$$

T_a: Fréchet differential of f at a

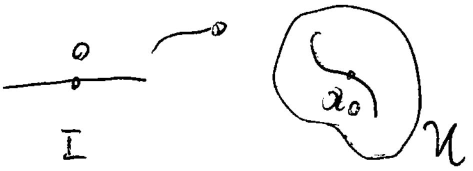
Example: $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, T_a is represented by the Jacobian matrix, (which is an $m \times n$ - matrix)

T.F.

Remark. The following observation will be important in the sequel

[everything $\mathcal{C}^\infty$, in order to fix ideas]

Let $\quad x = x(t) \in \mathcal{U}, \quad t \in I \quad$ I interval (containing 0)



$x(0) = x_0 \in \mathcal{U}$

Set $\quad \dot{x}(0) = h$

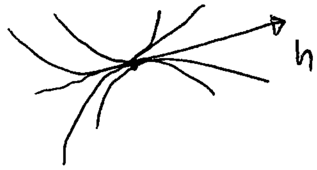
↗ velocity in 0

Set $\quad F = F(t) = f(x(t)) \quad (= (f \circ x)(t))$

Then $\quad (df|_{x_0})(h) = \dfrac{dF}{dt}(0)$

$$\left(= \sum_{i=1}^{n} f^{0}_{x_i} h_i \right) \qquad f^{0}_{x_i} = \frac{\partial f}{\partial x_i}(x_0)$$

__independently__ of $x = x(t)$, provided $\dot{x}(0) = h$ (fixed)



4. Integral

Let $V = C_c^0(\mathbb{R})$ (compactly supported continuous functions on $\mathbb{R}$)

[recall:
$\operatorname{supp} f = \overline{\{ x \in \mathbb{R} \,/\, f(x) \neq 0 \}}$ ~ closure]

Set

$$\int_{\mathbb{R}} : \quad V \ni f \longmapsto \int_{\mathbb{R}} f \quad \in \mathbb{R}$$

↑ Riemann integral

[notice that Riemann integration does not require a measure, the standard measure of parallelopodes being sufficient]

$\mu(P) := \prod_{i=1}^{n} (b_i - a_i)$ obvious notation

Then $\int_{\mathbb{R}}$ is a linear functional (continuous and positive, cf functional analysis course)

Aside:
The (measure theoretic) Riesz representation theorem tells us that $\int_{\mathbb{R}}$ is in fact integration with respect to the Lebesgue measure (that is $\int_{\mathbb{R}} f = \int_{\mathbb{R}} f(x)\, d\mu(x)$)

Lebesgue measure

★ Musical isomorphisms

Let $K = \mathbb{R}$, and let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean vector space, i.e. $\langle \cdot, \cdot \rangle$ is an inner product (scalar product):

[work in finite dimensions]

$$\langle \, , \, \rangle : \quad V \times V \longrightarrow \mathbb{R}$$

is a function fulfilling the following properties:

1. bilinearity

(linearity in both arguments

$$\langle \alpha v_1 + \beta v_2, w \rangle = \alpha \langle v_1, w \rangle + \beta \langle v_2, w \rangle \quad \text{etc}$$

2. symmetry

$$\langle v, w \rangle = \langle w, v \rangle \qquad \forall\, v, w \in V$$

3. positive definiteness

$\langle v, v \rangle \geq 0$ and equality holds if and only if $v = \underline{0}$

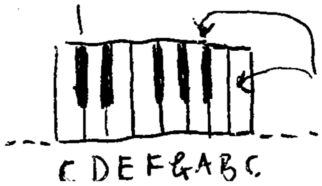
$\langle \, , \, \rangle$ induces specific isomorphisms (musical isomorphisms) between V and V^*

$\flat$: flat
$\sharp$: sharp

$A = LA$
$A^\sharp = B^\flat$ ($LA^\sharp = SI^\flat$)
$B = SI \dots$
$C^\sharp = D^\flat \dots$

$$\begin{array}{cccccc} & V & & V^* & & \\ & \Psi & & & & \\ \flat : & v & \longmapsto & v^\flat & = & \langle v, \cdot \rangle \end{array}$$

inner product against a fixed vector



$\flat$ is clearly injective

[$v^\flat = 0$ iff $\langle v, w \rangle = 0 \quad \forall\, w \in V$

$\Rightarrow$ in particular $\langle v, v \rangle = 0 \Rightarrow v = 0$
(positive definiteness)]

hence surjective ($N+R$), its inverse is called $\sharp$. This is also expressed by means of the Riesz representation theorem, in the following guise:

$\not\uparrow$ Let $(V, \langle\,,\rangle)$ be a Euclidean vector space, $\dim_{\mathbb{R}} V = n$. Let $l \in V^*$. Then, $\exists!\, u \in V$ such that, $\forall\, v \in V$, one has

$$l(v) = \langle u, v \rangle$$

[conversely, as we have already observed, $\forall\, u \in V$, the position $l_u(v) := \langle u, v \rangle$ defines a linear functional]

<u>Proof</u>. Let $(e_1, \dots e_n)$ be an orthonormal basis of V, i.e. $\langle e_i, e_j \rangle = \delta_{ij}$ $i = 1 .. n$. [Such a basis can be manufactured from any basis via the Gram-Schmidt procedure] . Then $v = \sum \alpha_i e_i$ (α_i uniquely determined)

Thus $l(v) = \sum_{i=1}^{n} \alpha_i\, l(e_i)$. Set $u_i = l(e_i)$

and $u = \sum_{i=1}^{n} u_i e_i$. Then u is the sought-for vector

Indeed: $\langle u, v \rangle = \langle \sum_i u_i e_i , \sum \alpha_j e_j \rangle$

$$= \sum_{i,j} u_i \alpha_j \underbrace{\langle e_i, e_j \rangle}_{\delta_{ij}} = \sum_{i=1}^{n} u_i \underset{\substack{\parallel \\ f(e_i)}}{\alpha_i} = l(v) \qquad \square$$

Notice that we employed bases...

We wish to be more explicit

Concretely, $\langle\ ,\ \rangle$ can be represented, given <u>any</u> basis $(e_1, \dots e_n)$, via a matrix $G = (g_{ij} = \langle e_i, e_j \rangle)$

In fact if $v = \sum v^i e_i \quad w = \sum w^j e_j$

notice this

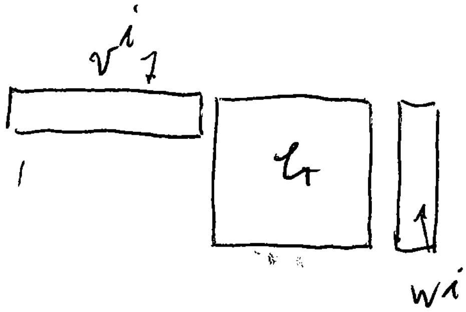
$$\langle v, w \rangle = \sum_{i,j} v^i w^j \langle e_i, e_j \rangle = \sum_{i,j} g_{ij} v^i w^j$$

with G symmetric and positive definite.

Conversely, given a basis in V and a symmetric, positive definite G, one defines an inner product via the above formula. The latter, in turn, can be rewritten as follows:

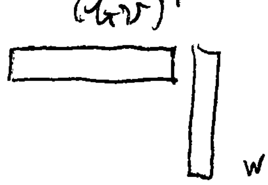
$$\langle v, w \rangle = v^T \cdot G \cdot w$$

(matrix products...)



But $v^T G w =$

$$v^T G^T w = (Gv)^T w = l_{Gv}(w)$$



where l_{Gv} is the linear functional corresponding to the row vector $(Gv)^T$

One sets $v_i := g_{ij} v^j$ ~ Einstein's summation convention

so $\langle v, w \rangle = v_i w^i$

Therefore $\flat : (v^i) \longmapsto (v_i)$

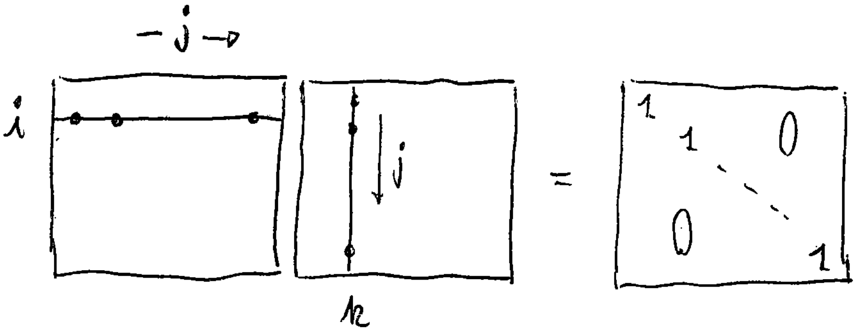
i.e. it <u>lowers</u> indices.

this explains the musical terminology

Let us visualise $\# = \flat^{-1}$: this will involve the inverse G^{-1} of G. Set $G^{-1} = (g^{ij})$

Then by definition $g^{ij} g_{jk} = \delta^i_k = \delta_{ik}$

again



$G^{-1} G = G G^{-1} = I$

Start from l, i.e. from a row vector $(v_i) = v^T$

components of l with respect to the dual basis $(e_1^* \dots e_n^*)$

One finds, successively:

$\left((A^{-1})^T = (A^T)^{-1} \equiv A^{-T} \right)$

$$l(v) = v^T w = v^T G^{-1} G w = v^T G^{-T} G w$$

$$= (G^{-1} v)^T G w \equiv \langle v^\#, w \rangle$$

$$v^\# = G^{-1} v$$

in components: $v^i = g^{ij} v_j$

... indices are <u>raised</u>

Summarizing: $\boxed{\begin{aligned} v_i &= g_{ij} v^j \\ v^i &= g^{ij} v_j \end{aligned}}$ $\begin{matrix} \flat \\ \# \end{matrix}$

$$v_i w^i = g_{ij} v^j w^i = g_{ij} w^i v^j = \dots = v^i w_i$$

observe

Example

Let $(\mathbb{R}^2, \langle\,,\rangle)$, with $\langle\,,\rangle$ represented, w.r. to the canonical basis, by

$$G = (g_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

recall:

$v_i = g_{ij} v^j$

$v^i = g^{ij} v_j$

Let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find $v^\flat$.

One has

$$v^\flat = (Gv)^T$$

$$= \left[\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]^T = \begin{pmatrix} 2 \\ 4 \end{pmatrix}^T = (2,4)$$

Now, given $\omega = (2, 4)$, find $\omega^\#$.

We have

caveat

$$\omega^\# = G^{-1}\omega^T = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as expected, since $\flat = \#^{-1}$, $\# = \flat^{-1}$.