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Master's Program in Applied Mathematics
First written test of Functional Analysis
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Solve the following problems. Justify your conclusions. Time: 120 min.

Pb 1. Let $a \in C([0,1] \times [0,1])$ and define $T : C([0,1]) \to C([0,1])$ by setting

$$T(u)(x) := \int_0^1 a(x,\sigma)u(\sigma)d\sigma, \quad x \in [0,1].$$

Show that T maps bounded sequences into relatively compact sequences.

Pb 2. Let $(a_n) \subset \mathbf{R}$ be a sequence such that $(a_n x_n) \in \ell^1$ for any $(x_n) \in \ell^2$. Show that $(a_n) \in \ell^2$.
(Hint: Consider the linear functional $T : (x_n) \mapsto \sum_{n=1}^{\infty} a_n x_n \ldots$ *S. Banach and H. Steinhaus may have something to tell.)*

Pb 3. Let $u \in L^2(\mathbf{R})$. Show that $T : \mathbf{R} \to L^2(\mathbf{R})$, $T(y) := u(\cdot - y)$ is continuous.

Pb 4. Let $Y = \{(x_n) \in \ell^1 : x_1 = x_3 = x_5 = \cdots = 0\}$ and let $\varphi \in Y'$. Prove that there exist infinitely many linear extensions $\Phi : \ell^1 \to \mathbf{R}$ of φ with $\|\Phi\|_{(\ell^1)'} = \|\varphi\|_{Y'}$.

Pb 5. For $\alpha \in (0,1]$, discuss the strong and weak convergence in $L^2(\mathbf{R}^2)$ of the sequence of functions

$$f_n(x,y) = n^\alpha e^{-n^2x^2 - n^2y^2}$$

Pb 6. Compute $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{\sin(n\sqrt{\sigma})}{2n\sigma + \sigma^3} d\sigma$.