4.6 Exercises - Part 2

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Exercise 5. Let R be a ring.

(a) An idempotent element $e \in R$ is called *primitive* if it is not a sum of two non zero orthogonal idempotents. Show that Re is indecomposable if and only if e is primitive.

(b) Find the decomposition in indecomposable summands of

(i) $M_2(\mathbb{C})$= the ring of 2×2 matrices with coefficients in $\mathbb{C}$,

(ii) the path algebra of the quiver $Q : \bullet \leftarrow \bullet \rightarrow \bullet \rightarrow \bullet$ over $\mathbb{C}$.

(c) Let E_i, $i = 1, \ldots, n$, be R-modules. Show: $\bigoplus_{i=1}^n E_i$ is injective if and only if E_i is injective for any $i = 1 \ldots n$.

(d) Let $f \in \mathrm{Hom}_R(L, M)$ be an essential monomorphism, and $g \in \mathrm{Hom}_R(M, N)$.

Show: if gf is a monomorphism, then so is g.

Exercise 6. (a) Write the representation $K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} K^2$ of $\mathbb{A}_2$ as a direct sum of two indecomposable representations.

(b) Find the injective envelope of the representation $0 \to K$ of $\mathbb{A}_2$.

(c) Given the path algebra $\Lambda = kQ$ of the quiver $Q : 1 \leftarrow 2 \rightarrow 3 \rightarrow 4$, compute the module Λe_2, its radical, and its socle.

Exercise 7. (a) Show: If X is a generating set of the $\mathbb{Z}$-module $\mathbb{Q}$, and $x \in X$, then $X \setminus \{x\}$ is a generating set of $\mathbb{Q}$ as well.

(b) Deduce from (a) that every finitely generated submodule of ${}_{\mathbb{Z}}\mathbb{Q}$ is superfluous.

(c) Conclude that $\mathrm{Rad}_{\mathbb{Z}}\, \mathbb{Q} = \mathbb{Q}$, and $\mathbb{Q}$ has no maximal submodules.

(d) Let M be a finitely generated left R-module iver a ring R. Show that any proper submodule $L < M$ is contained in a maximal submodule of M.

Exercise 8. Show that $\mathrm{Rad}({}_R R) = \{r \in R \mid 1 - xr \text{ has a left inverse for any } x \in R\}$. (Hint: Argue by contradiction, and use that $\mathrm{Rad}({}_R R)$ is the intersection of the annihilators of the simple left R-modules for $\supseteq$.)