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GAME THEORY and APPLICATIONS

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Games in normal form

There are given n-players. The set of all strategies (possible actions) of the i-th player is denoted by Σ_i. Each player choose an element $\sigma_i \in \Sigma_i$, $i = 1, 2, \ldots, n$. Then the mathematical expectation of the payoff function for the i-th player is given by $\pi_i(\sigma_1, \sigma_2, \ldots, \sigma_n)$, $i = 1, 2, \ldots, n$.

Definition

We say that the sets Σ_i, $i = 1, 2, \ldots, n$, and the functions $\pi_i : \Sigma_1 \times, \Sigma_2 \times \cdots \times \Sigma_n \to R$, $i = 1, 2, \ldots, n$, determine a game Γ in normal form.

Remark

Each player want to maximize his payoff function, but this depends on the choice of all players!

Nash equilibriom

Definition

Let the game Γ is determined by the sets Σ_i, $i = 1, 2, \ldots, n$, and the functions $\pi_i : \Sigma_1 \times, \Sigma_2 \times \cdots \times \Sigma_n \to R$, $i = 1, 2, \ldots, n$. It is said that the n-tuple

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Remark

Each player want to maximize his payoff function, but this depends on the choice of all players!

Nash equilibrium

Definition

Given a game Γ, a strategy n-tuple $(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)$ is said to be a Nash equilibrium if for any index $i = 1, 2, \ldots, n$, and any $\sigma_i \in \Sigma_i$,

$$\pi_i(\bar{\sigma}_1, \ldots, \bar{\sigma}_{i-1}, \sigma_i, \bar{\sigma}_{i+1} \ldots, \bar{\sigma}_n) \leq \pi(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n).$$

In other words, a strategy n-tuple $(\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)$ is said to be a Nash equilibrium if no player has a reason to change his strategy, assuming that none of the other players is going to change his strategy.

Zero-sum games

Definition

A game Γ is said to be zero-sum if

$$\Sigma_{i=1}^{n} \pi_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1} \ldots, \sigma_n) = 0$$

for any $\sigma_i \in \Sigma_i$, $i = 1, \ldots, n$.

In general, a zero-sum games represents a closed system: everything that someone wins must be lost by someone else.

Most parlor games are of the zero-sum type.

Two-person zero-sum games are called strictly competitive games.

Two-person zero-sum games

Let $(\bar{\sigma}_1, \bar{\sigma}_2) \in \Sigma_1 \times \Sigma_2$ be a Nash equilibrium. Then

$$\pi_1(\sigma_1, \bar{\sigma}_2) \leq \pi_1(\bar{\sigma}_1, \bar{\sigma}_2) \text{ for each } \sigma_1 \in \Sigma_1$$

and

$$\pi_2(\bar{\sigma}_1, \sigma_2) \leq \pi_2(\bar{\sigma}_1, \bar{\sigma}_2) \text{ for each } \sigma_2 \in \Sigma_2.$$

Because in this case $\pi_2(\sigma_1, \sigma_2) = -\pi_1(\sigma_1, \sigma_2)$, we obtain that

$$\pi_1(\sigma_1, \bar{\sigma}_2) \leq \pi_1(\bar{\sigma}_1, \bar{\sigma}_2) \leq \pi_1(\bar{\sigma}_1, \sigma_2)$$

for each $\sigma_1 \in \Sigma_1$ and each $\sigma_2 \in \Sigma_2$, i.e. $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a saddle point for the function π_1.

Two-person zero-sum games

We have obtained that the first player want to maximize $\pi_1(\sigma_1, \bar{\sigma}_2)$ over Σ_1, while the second player want to minimize $\pi_1(\bar{\sigma}_1, \sigma_2)$ over Σ_2.

Let us assume that first player is omniscient and he can guess correctly the action $\sigma_2 \in \Sigma_2$ of the second player. Then the first player will choose $\bar{\sigma}_1 \in \Sigma_1$ so that

$$\pi_1(\bar{\sigma}_1, \sigma_2) = \max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \sigma_2).$$

For simplicity, we assume here and further that these "max" and "min" exist.

But then it is natural the second player to choose $\bar{\sigma}_2 \in \Sigma_2$ so that

$$\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \bar{\sigma}_2) = \min_{\sigma_2 \in \Sigma_2} \left(\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \sigma_2) \right)$$

Two-person zero-sum games

Analogously it is natural for the first player to choose $\bar{\sigma}_1 \in \Sigma_1$ so that

$$\min_{\sigma_2 \in \Sigma_2} \pi_1(\bar{\sigma}_1, \sigma_2) = \max_{\sigma_1 \in \Sigma_1} \left(\min_{\sigma_2 \in \Sigma_2} \pi_1(\sigma_1, \sigma_2) \right)$$

Definition

It is natural to define

$$\text{“gain-floor”} \;\; v_I := \max_{\sigma_1 \in \Sigma_1} \left(\min_{\sigma_2 \in \Sigma_2} \pi_1(\sigma_1, \sigma_2) \right)$$

and

$$\text{“loss-ceiling”} \;\; v_{II} := \min_{\sigma_2 \in \Sigma_2} \left(\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \sigma_2) \right).$$

The meaning of v_I and v_{II} is: the first player should not win less than v_I and the second player should not loss more than v_{II}.

Two-person zero-sum games

Lemma 1.

We have that $v_I \leq v_{II}$.

Proof:

Let us fix arbitrary elements $\hat{\sigma}_1 \in \Sigma_1$ and $\hat{\sigma}_2 \in \Sigma_2$. Clearly, we have that

$$\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \hat{\sigma}_2) \geq \pi_1(\hat{\sigma}_1, \hat{\sigma}_2) \geq \min_{\sigma_2 \in \Sigma_2} \pi_1(\hat{\sigma}_1, \sigma_2), \text{ i.e.}$$

$$\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \hat{\sigma}_2) \geq \min_{\sigma_2 \in \Sigma_2} \pi_1(\hat{\sigma}_1, \sigma_2),$$

Since the right-hand side does not depend on $\bar{\sigma}_1$ and the left-hand side does not depend on $\bar{\sigma}_2$, we obtain that

Proof: (continuation)

$$\min_{\sigma_2 \in \Sigma_2} \left(\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \hat{\sigma}_2) \right) \geq \max_{\sigma_1 \in \Sigma_1} \left(\min_{\sigma_2 \in \Sigma_2} \pi_1(\hat{\sigma}_1, \sigma_2) \right),$$

i.e. $v_{II} \geq v_I$. This completes the proof.

Definition.

If $v_I = v_{II}$, the common number $v := v_I = v_{II}$ is said to be value of the game Γ.

Two-person zero-sum games

Lemma 2.

A game Γ has a value, if and only if, it has a saddle point.

Proof. (sufficiency)

Let the game Γ has a saddle point, i.e. there exists $(\bar{\sigma}_1, \bar{\sigma}_2) \in \Sigma_1 \times \Sigma_2$ so that

$$\pi_1(\sigma_1, \bar{\sigma}_2) \leq \pi_1(\bar{\sigma}_1, \bar{\sigma}_2) \leq \pi_1(\bar{\sigma}_1, \sigma_2)$$

for each $\sigma_1 \in \Sigma_1$ and each $\sigma_2 \in \Sigma_2$. Then

$$\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \bar{\sigma}_2) \leq \pi_1(\bar{\sigma}_1, \bar{\sigma}_2) \leq \min_{\sigma_2 \in \Sigma_2} \pi_1(\bar{\sigma}_1, \sigma_2)$$

and hence

Proof. (sufficiency: continuation)

$$\min_{\sigma_2 \in \Sigma_2} \left(\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \bar{\sigma}_2) \right) \leq \max_{\sigma_1 \in \Sigma_1} \left(\min_{\sigma_2 \in \Sigma_2} \pi_1(\bar{\sigma}_1, \sigma_2) \right)$$

i.e. $v_{II} \leq v_I$. We have already prove that $v_I \leq v_{II}$. Hence $v_I = v_{II}$ and the game Γ has a value. This completes the proof of the sufficiency.

Proof. (necessity)

Let the game Γ has a value

$$v = \min_{\sigma_2 \in \Sigma_2} \left(\max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \bar{\sigma}_2) \right) \leq \max_{\sigma_1 \in \Sigma_1} \left(\min_{\sigma_2 \in \Sigma_2} \pi_1(\bar{\sigma}_1, \sigma_2) \right).$$

Two-person zero-sum games

Proof. (necessity: continuation)

According our simplifying assumption, there exist $\bar{\sigma}_1 \in \Sigma_1$ and $\bar{\sigma}_2 \in \Sigma_2$ so that

$$v = \min_{\sigma_2 \in \Sigma_2} \pi_1(\bar{\sigma}_1, \sigma_2) = \max_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \bar{\sigma}_2),$$

and hence

$$\pi_1(\sigma_1, \bar{\sigma}_2) \leq v \leq \pi_1(\bar{\sigma}_1, \sigma_2) \text{ for each } \sigma_1 \in \Sigma_1 \text{ and each } \sigma_2 \in \Sigma_2.$$

It follows from here that

$$\pi_1(\sigma_1, \bar{\sigma}_2) \leq v = \pi_1(\bar{\sigma}_1, \bar{\sigma}_2) \leq \pi_1(\bar{\sigma}_1, \sigma_2)$$

for each $\sigma_1 \in \Sigma_1$ and each $\sigma_2 \in \Sigma_2$, i.e. $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a saddle point for the function π_1. This completes the proof of the necessity.

Two-person zero-sum games

Definition

A mixed strategy is a probability distribution on the set of his pure strategies.

Let the first player has m pure strategies. Then the set X of its mixed strategies consists of all vectors $x = (x_1, x_2, \dots, x_m)$ whose components satisfy

$$\sum_{i=1}^{m} x_i = 1 \text{ and } x_i \geq 0,\ i = 1, \dots, m.$$

Analogously, if the second player has n pure strategies. Then the set Y of its mixed strategies consists of all vectors $y = (y_1, y_2, \dots, y_n)$ whose components satisfy

$$\sum_{j=1}^{n} y_j = 1 \text{ and } y_j \geq 0,\ j = 1, \dots, n.$$

Two-person zero-sum games

Let us assume that players I and II are playing a zero-sum game determined by the matrix A. If I chooses the mixed strategy x, and II chooses y, then the expected payoff is

$$P(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j = \sum_{i=1}^{m} x_i \left(\sum_{j=1}^{n} a_{ij}y_j\right) = \sum_{j=1}^{n} y_j \left(\sum_{j=1}^{n} a_{ij}x_i\right)$$

$$\text{i.e. } P(x,y) = \sum_{i=1}^{m} x_i P(i,y) = \sum_{j=1}^{n} y_j P(x,j).$$

Here i in the expression $P(i,y)$ denotes the i-th pure strategy of I. Analogously, j in the expression $P(x,j)$ denotes the j-th pure strategy of II.

Lemma 3.

Let $(x_1, y_1) \in X \times Y \ni (x_2, y_2)$ be saddle points of the payoff-function P. Then (x_1, y_2) and (x_2, y_1) are also saddle points ofthe payoff-function P.

Proof.

The definition of a saddle point implies that

$$P(x_2, y_1) \leq P(x_1, y_1) \leq P(x_2, y_2)$$

and

$$P(x_1, y_2) \leq P(x_2, y_2) \leq P(x_2, y_1).$$

These inequalities imply that

$$P(x_2, y_1) = P(x_1, y_1) = P(x_2, y_2) = P(x_2, y_1).$$

Proof. (continuation)

Let x and y be arbitrary mixed strategies from X and Y, respectively. Then we have that

$$P(x, y_1) \leq P(x_1, y_1) = P(x_2, y_1) = P(x_2, y_2) \leq P(x_2, y),$$

i.e. (x_1, y_2) is a also saddle point of the payoff-function P. Analogously, one can prove that (x_2, y_1) is also a saddle point of the payoff-function P.

Lemma 4.

Let $(\bar{x}, \bar{y}) \in X \times Y$ be a saddle point of the payoff-function P. If $\bar{x}_{i_0} > 0$, then $P(i_0, \bar{y}) = P(\bar{x}, \bar{y})$. Also, if $\bar{y}_{j_0} > 0$, then $P(\bar{x}, j_0) = P(\bar{x}, \bar{y})$.

Proof.

Let us assume the contrary, i.e. $P(i_0, \bar{y}) < P(\bar{x}, \bar{y})$. Since $(\bar{x}, \bar{y}) \in X \times Y$ is a saddle point of the payoff-function P, we have that

$$P(i, \bar{y}) \leq P(\bar{x}, \bar{y}) \text{ for each } i = 1, 2, \ldots, m, \text{ with } i \neq i_0.$$

Multiplying the both sides of these inequalities by $\bar{x}_i$, we obtain that

$$x_i P(i, \bar{y}) \leq x_i P(\bar{x}, \bar{y}).$$

After adding of all these m inequalities, we obtain that

$$\sum_{i=1}^{m} \bar{x}_i P(i, \bar{y}) < \sum_{i=1}^{m} \bar{x}_i P(\bar{x}, \bar{y}), \text{ i.e. } P(\bar{x}, \bar{y}) < P(\bar{x}, \bar{y}).$$

The obtained contradiction shows that our assumption is wrong, and hence $P(i_0, \bar{y}) = P(\bar{x}, \bar{y})$. Analogously, one can prove that $P(\bar{x}, j_0) = P(\bar{x}, \bar{y})$. This completes the proof.

Remark.

Let $(\bar{x}, \bar{y}) \in X \times Y$ be a saddle point of the payoff-function P. Then $P(\bar{x}, \bar{y})$ is a value of the corresponding zero-sum game.

Lemma 5.

Let $(\bar{x}, \bar{y}) \in X \times Y$ be a saddle point of the payoff-function P determined by the matrix $A = (a_{ij})_{m \times n}$. Then $(\bar{x}, \bar{y})$ is a saddle point for the zero-sum game determined by the matrix $B = (b_{ij})_{m \times n}$ with $b_{ij} = \alpha a_{ij} + \beta$, where $\alpha > 0$. Moreover, $\alpha P(\bar{x}, \bar{y}) + \beta$ is the value of the zero-sum game determined by the matrix B.

Two-person zero-sum games

Proof.

Let us denote by P_A and P_B the payoff-functions of the zero-sum games generated by the matrices A and B, respectively. Then

$$P_B(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} b_{ij} x_i y_j = \sum_{i=1}^{m} \sum_{j=1}^{n} (\alpha a_{ij} + \beta) x_i y_j =$$

$$= \alpha \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} + \beta \sum_{i=1}^{m} x_i \sum_{j=1}^{n} y_j = \alpha P_A(x, y) + \beta$$

Because $(\bar{x}, \bar{y}) \in X \times Y$ is a saddle point of the payoff-function P_A, the following inequalities hold true:

$$P_A(x, \bar{y}) \leq P_A(\bar{x}, \bar{y}) \leq P_A(\bar{x}, y)$$

for each $x \in X$ and $y \in Y$.

Proof of Lemma 5 (continuation).

These inequalities imply that

$$\alpha P_A(x, \bar{y}) + \beta \leq \alpha P_A(\bar{x}, \bar{y}) + \beta \leq \alpha P_A(\bar{x}, y) + \beta,$$

i.e.

$$P_B(x, \bar{y}) \leq P_B(\bar{x}, \bar{y}) \leq P_B(\bar{x}, y)$$

for each $x \in X$ and $y \in Y$. Hence $(\bar{x}, \bar{y})$ is a saddle point of the payoff-function P_B and $P_B(\bar{x}, \bar{y})$ is its value.

Remark.

Let $A = (a_{ij})_{m \times n}$ be an arbitrary matrix and $M := 1 + \max\limits_{1 \leq i \leq m, 1 \leq j \leq n} |a_{ij}|$. We set $B = (b_{ij})_{m \times n}$ with $b_{ij} = a_{ij} + M$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Then the all elements of the matrix B are positive, and hence, its value is also positive.

Lemma 6.

Let $(\bar{x}, \bar{y}) \in X \times Y$ be a saddle point of the payoff-function P. If $P(i_0, \bar{y}) < P(\bar{x}, \bar{y})$, then $\bar{x}_{i_0} = 0$. Also, if $P(\bar{x}, j_0) > P(\bar{x}, \bar{y})$, then $\bar{y}_{j_0} = 0$.

Proof.

Let us assume that $\bar{x}_{i_0} > 0$. According to Lemma 4 we obtain that $P(i_0, \bar{y}) = P(\bar{x}, \bar{y})$. This contradiction shows that $\bar{x}_{i_0} = 0$. Analogously, one can prove that $\bar{y}_{j_0} = 0$.

Proposition 1.

Let S be a closed subset of R^n and let $x \notin S$. Then there exists point $y \in S$ so that

$$0 < \|x - y\| = \min(\|x - s\| : \ s \in S).$$

Proof of Proposition 1.

Let $d := \inf(\|x - s\| : \; s \in S)$ and let n be an arbitrary positive integer. Because $d + \frac{1}{n} > d$ there exists point $y_n \in S$ so that $d \leq \|x - y_n\| < d + \frac{1}{n}$. Because

$$\|y_n\| \leq \|x - y_n\| + \|x\| \leq d + 1 + \|x\|,$$

the sequence $\{y_n\}_{n=1}^{\infty}$ is bounded, and hence there exists a convergent subsequence $\{y_{n_k}\}_{k=1}^{\infty}$ tending to y as k tends to infinity. Taking a limit in the inequalities $d \leq \|x - y_{n_k}\| < d + \frac{1}{n_k}$ as $k \to \infty$, we obtain that $d \leq \|x - y\| \leq d$. This completes the proof.

Two-person zero-sum games

We denote by $\langle a, b\rangle := \sum_{i=1}^{n} a_i b_i$ the scalar product of two vectors $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$.

Separability theorem:

Let S be a closed convex subset of R^n and let the point x does not belong to S. Then there exist a non-zero vector p and a real number q such that $\langle p, x\rangle = q$ and $\langle p, s\rangle > q$ for each $s \in S$.

Proof.

Let $y \in S$ be a point of S such that $\|x - y\| =$
$= \min(\|x - s\| : \ s \in S)$. We set $p := y - x$ and $q := \langle p, x\rangle$.
Then

$$\langle p, y\rangle - q = \langle p, y\rangle - \langle p, x\rangle = \langle p, y - x\rangle = \|p\|^2 > 0.$$

Proof (continuation).

Let us assume that there exists a point $s \in S$ so that $\langle p, s\rangle \leq q$. Then

$$r := \langle p, s-y\rangle = \langle p, s\rangle - \langle p, y\rangle < q - q = 0.$$

For each $\varepsilon \in (0,1)$ we set $s_\varepsilon := (1-\varepsilon)y + \varepsilon s$. The convexity of S implies that $x_\varepsilon \in S$.
On can directly checked that

$$\|x - x_\varepsilon\|^2 = \|(x-y) + \varepsilon(s-y)\|^2 =$$

$$\|x-y\|^2 + 2\varepsilon \langle y-x, s-y\rangle + \varepsilon^2\|s-y\|^2 =$$

$$= \|x-y\|^2 + \varepsilon\left(2r + \varepsilon\|s-y\|^2\right) < \|x-y\|^2$$

for each sufficiently small $\varepsilon > 0$ (because $r < 0$). The obtained contradiction completes the proof.

Theorem of the Alternative for matrices

Let $A = (a_{ij})_{m \times n}$ be an arbitrary real matrix. Then either (i) or (ii) must hold:

(i) The point $0 \in R^m$ belongs to the convex hull C of the vectors $a_1, a_2, \ldots, a_n$, $e_1, e_2, \ldots, e_m$, where $a_j := (a_{1j}, a_{2j}, \ldots, a_{mj})$, $j = 1, 2, \ldots, n$, and $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_m = (0, 0, \ldots, 1)$.

(ii) There exists a vector $p = (p_1, p_2, \ldots, p_m)$ such that $\sum_{i=1}^{m} p_i = 1$ and $p_i > 0$ for each $i = 1, 2, \ldots, m$, and $\langle p, a_j \rangle > 0$ for each $j = 1, 2, \ldots, n$.

Proof.

Let us assume that $0 \notin C$. Applying the Separability theorem, we obtain that there exist a non-zero vector p and a number q such that $q = \langle p, 0 \rangle = 0$, $p_i = \langle p, e_i \rangle > 0$ for each $i = 1, 2, \dots, m$, and $\langle p, a_j \rangle > 0$ for each $j = 1, 2, \dots, n$, i.e. (ii) holds true.

The min-max theorem (von Neuman and Morgenstern)

Let $A = (a_{ij})_{m \times n}$ be an arbitrary real matrix. Then the zero-sum game determined by the matrix A has a value, i.e. $v_I^A = v_{II}^A$.

Proof of the min-max theorem .

According to the Theorem of the Alternative for matrices, either (i) or (ii) must hold.
If (i) is fulfilled, then there exists $\alpha_1 \geq 0, \alpha_2 \geq 0, \ldots, \alpha_n \geq 0$ and $\beta_1 \geq 0, \beta_2 \geq 0, \ldots, \beta_m \geq 0$ such that

$$\sum_{j=1}^{n} \alpha_j + \sum_{i=1}^{m} \beta_i = 1$$

and

$$\sum_{j=1}^{n} \alpha_j a_j + \sum_{i=1}^{m} \beta_i e_i = 0,$$

Proof (continuation)

i.e.

$$\sum_{j=1}^{n} \alpha_j a_{ij} + \beta_i = 0, \ i = 1, 2, \ldots, m.$$

If all $\alpha_j = 0$, $j = 1, 2, \ldots, n$, then all $\beta_i = 0$, $i = 1, 2, \ldots, m$, and hence their sum can not be qual to 1. The obtained contradiction shows that at least one α_j must be positive. We set

$$\bar{y}_j := \frac{\alpha_j}{\sum_{k=1}^{n} \alpha_k} \geq 0.$$

Proof (continuation)

Clearly, each $\bar{y}_j$, $j = 1, 2, \dots, n$, is well defined, $\sum_{j=1}^{n} \bar{y}_j = 1$ and

$$\sum_{j=1}^{n} \alpha_j a_{ij} = -\beta_i \le 0, \;\; i = 1, 2, \dots, m, \;\; \text{i.e.} \;\; P^A(i, \bar{y}) \le 0.$$

This implies that for each mixed strategy x of the first player

$$P^A(x, \bar{y}) = \sum_{i=1}^{m} x_i P^A(i, \bar{y}) \le 0, \;\; \text{and hence} \;\; \max_{x \in X} P^A(x, \bar{y}) \le 0.$$

Thus we obtain that $v_{II} = \min_{y \in Y} \max_{x \in X} P^A(x, y) \le 0$.

Two-person zero-sum games

Proof (continuation)

Suppose, instead, that (ii) holds true, i.e. there exists a vector $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ such that $\sum_{i=1}^{m} \bar{x}_i = 1$ and $\bar{x}_i > 0$ for each $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} \bar{x}_i a_{ij} = \langle \bar{x}, a_j \rangle > 0$ for each $j = 1, 2, \ldots, n$. Then $\bar{x}$ can be considered as a mixed strategy of the I player and $P^A(\bar{x}, j) > 0$. This implies that for each mixed strategy y of the second player

$$P^A(\bar{x}, y) = \sum_{j=1}^{n} y_j P^A(\bar{x}, j) > 0, \text{ and hence } \min_{y \in Y} P^A(\bar{x}, y) > 0.$$

Thus we obtain that $v_I = \max_{x \in X} \min_{y \in Y} P^A(x, y) > 0$.

Proof (continuation)

We know that $v_I^A \leq v_{II}^A$. Let us assume that $v_I^A < v_{II}^A$. We consider the matrix $B := (b_{ij})_{m \times n}$ with $b_{ij} := a_{ij} - \dfrac{v_I^A + v_{II}^A}{2}$. Then

$v_I^B = v_I^A - \dfrac{v_I^A + v_{II}^A}{2} = \dfrac{v_I^A - v_{II}^A}{2} < 0$. Analogously,

$v_{II}^B = v_{II}^A - \dfrac{v_I^A + v_{II}^A}{2} = -\dfrac{v_I^A - v_{II}^A}{2} > 0$. But this is impossible. The obtained contradiction shows that $v_I^A = v_{II}^A$ and completes the proof of the Min-max theorem.

Lemma 6.

Let $\bar{x}$ and $\bar{y}$ be mixed strategies of players I and II, and v be a real number such that

$$P(i, \bar{y}) \leq v \leq P(\bar{x}, j), \;\; i = 1, 2, \ldots, m, \;\; j = 1, 2, \ldots, n.$$

Then v is the value of the game and the couple $(\bar{x}, \bar{y})$ - a Nash equilibrium.

Proof.

Let x and y are arbitrary mixed strategies of players I and II. Then one can check that

$$P(x, \bar{y}) = \sum_{i=1}^{m} x_i P(i, \bar{y}) \leq \sum_{i=1}^{m} x_i v = v =$$

$$= \sum_{j=1}^{n} y_j v = \sum_{j=1}^{n} y_j P(\bar{x}, j) = P(\bar{x}, y).$$

Without loss of generality, we may think that the value v of a game determined by the matrix $A = (a_{ij})_{m\times n}$ is positive (for, example, if all elements $a_{ij} > 0$). Let $\bar{v}_x$ and $\bar{x}$ ($\bar{v}_y$ and $\bar{y}$) be solutions of the following linear problems

$v_y \to \min$	$v_x \to \max$
$P(i,y) \le v_y,\ \ i = 1,2,\dots,m$	$P(x,j) \ge v_x,\ \ j = 1,2,\dots,n$
$\sum_{j=1}^{n} y_j = 1$	$\sum_{i=1}^{m} x_i = 1$
$y_j \ge 0,\ \ j = 1,2,\dots,n$	$x_i \ge 0,\ \ i = 1,2,\dots,m$

These problems can be written as follows

Two-person zero-sum games

$$
\begin{array}{l|l}
v_y \to \min & v_x \to \max \\
\sum_{j=1}^{n} a_{ij} y_j \le v_y, \;\; i = 1, 2, \dots, m & \sum_{i=1}^{m} a_{ij} x_i \ge v_x, \;\; j = 1, 2, \dots, n \\
\sum_{j=1}^{n} y_j = 1 & \sum_{i=1}^{m} x_i = 1 \\
y_j \ge 0, \; j = 1, 2, \dots, n & x_i \ge 0, \;\; i = 1, 2, \dots, m
\end{array}
$$

These two linear problems are dual. If $(v_x, \bar{x})$ and $(v_y, \bar{y})$ are the solutions of the first and second problem, respectively, then $v_x = v_y = v$ is the value of the game and the couple $(\bar{x}, \bar{y})$ is a Nash equilibrium.

If we set $p_i = x_i/v$, $i = 1, 2, \ldots, m$, and $q_j = y_j/v$, $j = 1, 2, \ldots, n$, then we can write the above problems as follows

$$\sum_{i=1}^{m} q_j \to \max$$

$$\sum_{j=1}^{n} a_{ij} q_j \leq 1, i = 1, 2, \ldots, m$$

$$q_j \geq 0, \ j = 1, 2, \ldots, n$$

$$\sum_{i=1}^{m} p_i \to \min$$

$$\sum_{i=1}^{m} a_{ij} p_i \geq 1, \ j = 1, 2, \ldots, n$$

$$p_i \geq 0, \ i = 1, 2, \ldots, m$$

Two-person zero-sum games

Zero-sum games determined by a matrix of type 2×2

Let us consider a game determined by the following matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let us assume that this game has no Nash equilibrium in pure strategies. Let $(\bar{x}, \bar{y})$ be a Nash equilibrium with $\bar{x} = (1, 0)$ and $\bar{y} = (\bar{y}_1, \bar{y}_2)$, where $\bar{y}_1 > 0$, $\bar{y}_2 > 0$ and $\bar{y}_1 + \bar{y}_2 = 1$. Then $P(x, \bar{y}) \leq P(\bar{x}, \bar{y}) \leq P(\bar{x}, y)$. Clearly, $v = P(\bar{x}, \bar{y}) = a\bar{y}_1 + b\bar{y}_2$. We set $y = (1, 0)$ and $y = (0, 1)$, and obtain that $a\bar{y}_1 + b\bar{y}_2 \leq \min(a, b)$. This implies that $a = b = v$. Also, we have that $v \geq P(2, \bar{y}) = c\bar{y}_1 + d\bar{y}_2 \geq \min(c, d)$. If $c = \min(c, d)$, we obtain that $((1.0), (1, 0))$ is a Nash equilibrium in pure strategies. If $d = \min(c, d)$, we obtain that $((1.0), (0, 1))$ is a Nash equilibrium in pure strategies.

But, according to our assumption, the game has no Nash equilibrium in pure strategies. Hence the assumption that $(\bar{x}, \bar{y})$ be a Nash equilibrium with $\bar{x} = (1, 0)$ and $\bar{y} = (\bar{y}_1, \bar{y}_2)$, where $\bar{y}_1 > 0$, $\bar{y}_2 > 0$ and $\bar{y}_1 + \bar{y}_2 = 1$, is not possible. Analogously it can be proved that it is not possible $(\bar{x}, \bar{y})$ to be a Nash equilibrium with $\bar{x} = (0, 1)$ and $\bar{y} = (\bar{y}_1, \bar{y}_2)$, where $\bar{y}_1 > 0$, $\bar{y}_2 > 0$ and $\bar{y}_1 + \bar{y}_2 = 1$. This shows that whenever a zero-sum game has no a Nash equilibrium in pure strategies, then both components of the Nash equilibrium are mixed strategies.

Reducing zero-sum games

Let us consider a zero-sum game determined by the matrix $A = (a_{ij})_{m\times n}$. Let the i_0-pure strategy of I is dominated by a convex combination of the remainder pure strategies of the I player, i.e.

$$a_{i_0 j} \leq \sum_{i=1, i\neq i_0}^{m} \alpha_i a_{i,j} \text{ for each } j = 1, 2, \ldots, n,$$

where all number α_i are nonnegative and

$$\sum_{i=1, i\neq i_0}^{m} \alpha_i = 1.$$

Two-person zero-sum games

We denote by A' the matrix obtained from the matrix A be deleting the i_0-raw. Let $(\bar{x}', \bar{y}')$ with $\bar{x}' = (\bar{x}'_1, \bar{x}'_2, \dots, \bar{x}'_{i_0-1}, \bar{x}'_{i_0+1}, \dots, \bar{x}'_m)$ be a Nash equilibrium for the game determined by the matrix A'. Then $(\bar{x}, \bar{y}')$ is a Nash equilibrium for the game determined by the matrix A, where $\bar{x} = (\bar{x}'_1, \bar{x}'_2, \dots, \bar{x}'_{i_0-1}, 0, \bar{x}'_{i_0+1}, \dots, \bar{x}'_m)$.

Because $(\bar{x}', \bar{y}')$ is a Nash equilibrium for the game determined by the matrix A', the following inequalities hold true:

$$P^{A'}(x', \bar{y}') \leq v = P^{A'}(\bar{x}', \bar{y}') \leq P^{A'}(\bar{x}', y'), \text{ for } x' \in X \text{ and } y \in Y,$$

where

$$P^{A'}(x', y') = \sum_{i=1, i \neq i_0}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j.$$

In particular, we have that

$$P^{A'}(i,\bar{y}') \leq v \leq P^{A'}(\bar{x}',j),\ i=1,\dots,i_0-1,i_0+1,\dots,m,\ j=1,\dots,n.$$

i.e.

$$\begin{aligned} \sum_{j=1}^{n} a_{ij}\bar{y}_j \leq v, \qquad & i=1,\dots,i_0-1,i_0+1,\dots,m, \\ \sum_{i=1,i\neq i_0}^{m} a_{ij}\bar{x}_i \geq v, \quad & j=1,\dots,n \end{aligned} \tag{1}$$

Since the vector $\alpha := (\alpha_1,\alpha_2,\dots,\alpha_{i_0-1},\alpha_{i_0+1},\dots,\alpha_m)$ can be considered as a mixed strategy of I, we have that $P^{A'}(\alpha,\bar{y}') \leq v$, i.e.

$$\sum_{i=1, i \neq i_0}^{m} \sum_{j=1}^{n} a_{ij} \alpha_i \bar{y}_j \leq v.$$

From here (taking into account that the i_0-pure strategy is dominated by a convex combination of the remainder pure strategies of the I player), it follows that

$$\sum_{j=1}^{n} a_{i_0 j} \bar{y}_j \leq \sum_{j=1}^{n} \sum_{i=1, i \neq i_0}^{m} a_{ij} \alpha_i \bar{y}_j \leq v.$$

Applying this inequality and 1, we complete the proof taking into account Lemma 6.

Analogously, let us assume that the j_0-pure strategy of II is dominated by a convex combination of the remainder pure strategies of the II player, i.e.

$$a_{ij_0} \geq \sum_{j=1, j \neq j_0}^{n} \beta_j a_{i,j} \text{ for each } i = 1, 2, \ldots, m,$$

where all number β_j are nonnegative and

$$\sum_{j=1, j \neq j_0}^{n} \beta_j = 1.$$

We denote by A' the matrix obtained from the matrix A be deleting the j_0-column. Let $(\bar{x}', \bar{y}')$ with $\bar{y}' = (\bar{y}'_1, \bar{y}'_2, \ldots, \bar{y}'_{j_0-1}, \bar{y}'_{j_0+1}, \ldots, \bar{y}'_n)$ be a Nash equilibrium for the game determined by the matrix A'. Then $(\bar{x}, \bar{y}')$ is a Nash equilibrium for the game determined by the matrix A, where $\bar{y}' = (\bar{y}'_1, \bar{y}'_2, \ldots, \bar{y}'_{j_0-1}, 0, \bar{y}'_{j_0+1}, \ldots, \bar{y}'_n)$.

$2 \times n$ and $m \times 2$ games

We consider first a zero-sum game determined by a the $2 \times n$ -matrix A. If v is its value, then

$$v = \max_{x \in X} \min_{y \in Y} \sum_{i=1}^{2} \sum_{j=1}^{n} a_{ij} x_i y_j = \min_{y \in Y} \max_{x \in X} \sum_{i=1}^{2} \sum_{j=1}^{n} a_{ij} x_i y_j .$$

Clearly,

$$\max_{x \in X} \min_{j=1,2,\ldots,n} \sum_{i=1}^{2} \sum_{j=1}^{n} a_{ij} x_i \geq v .$$

Two-person zero-sum games

On the other hand

$$\sum_{j=1}^{n}\sum_{i=1}^{2} a_{ij}x_i y_j = \sum_{j=1}^{n} y_j \sum_{i=1}^{2} a_{ij}x_i \geq$$

$$\geq \sum_{j=1}^{n} y_j \min_{j=1,2,\ldots,n} \sum_{i=1}^{2} a_{ij}x_i = \min_{j=1,2,\ldots,n} \sum_{i=1}^{2} a_{ij}x_i,$$

From here, it follows

$$\min_{y\in Y} \sum_{j=1}^{n}\sum_{i=1}^{2} a_{ij}x_i y_j \geq \min_{j=1,2,\ldots,n} \sum_{i=1}^{2} a_{ij}x_i,$$

and hence

Two-person zero-sum games

$$v = \max_{x \in X} \min_{y \in Y} \sum_{i=1}^{2} \sum_{j=1}^{n} a_{ij} x_i y_j \geq \max_{x \in X} \min_{j=1,2,\ldots,n} \sum_{i=1}^{2} a_{ij} x_i .$$

In such a way, we obtain that

$$v = \max_{x \in X} \min_{j=1,2,\ldots,n} \sum_{i=1}^{2} a_{ij} x_i .$$

Since $X = \{(x_1, x_2) : \; x_1 = \alpha, \; x_2 = 1 - \alpha, \; \alpha \in [0,1]\}$, we obtain that v is the maximum (with respect to α) of the minimum of n linear functions (depending on α). So we can plot these function and to find this maximum graphically.

$2 \times n$ and $m \times 2$ games

We consider next a zero-sum game determined by a the $m \times 2$ -matrix A. If v is its value, then

$$v = \min_{y \in Y} \max_{i=1,2,\dots,m} \sum_{j=1}^{2} a_{ij} y_j .$$

Since $Y\{(y_1, y_2) : \ y_1 = \beta, \ y_2 = 1 - \beta, \ \beta \in [0,1]\}$, we obtain that v is the minimum (with respect to β) of the maximum of m linear functions (depending on β). So we can plot these function and to find this minimum graphically.

Two-person zero-sum games

Symmetric games

A square matrix $A = (a_{ij})_{n \times n}$ is said to be skew-symmetric if $a_{ji} = -a_{ij}$ for all $i, j = 1, \ldots, n$. A zero-sum game is said to be symmetric if its matrix is skew-symmetric.

Theorem.

The value of a symmetric game is zero. If $(\bar{x}, \bar{y})$ is a Nash equilibrium, then $(\bar{x}, \bar{x})$ and $(\bar{y}, \bar{y})$ are also Nash equilibriums.

Proof.

Because the matrix A is skew-symmetric, $A^T = -A$. Let x be an arbitray mixed strategy. Then $x^T A x = (x^T A x)^T =^T A^T x =$ $= -x^T A x$, and hence $x^T A x = 0$.

Proof (continuation.

Then

$$0 \leq \max_{x \in X} \min_{y \in Y} x^T A x = v = \min_{y \in Y} \max_{x \in X} x^T A x \leq 0$$

which implies that $v = 0$.
Let j be an arbitrary index. Then $P(\bar{x}, j) \geq 0$, i.e.

$$\sum_{i=1}^{n} a_{ij} \bar{x}_i \geq 0, \text{ i.e. } -\sum_{i=1}^{n} a_{ji} \bar{x}_i \geq 0, \text{ i.e. } \tag{2}$$

$$\sum_{j=1}^{n} a_{ij} \bar{x}_j \leq 0, \text{ i.e. } P(i, \bar{x}) \leq 0.$$

The last inequality, (2) and Lemma 6 complete the proof.

Theorem.

A symmetric game has no a Nash equilibrium in pure strategies, if and only if one of the following assertions holds true:
(i) each column has at least one positive element;
(ii) each raw has at least one negative element.

Proof. (sufficiency)

We shall consider only (i). The case (ii) can be studied in the same way. Let (k, k) be a Nash equilibrium in pure strategies and let $a_{ik} > 0$. Then

$$0 < a_{ik} = P(i, k) \leq v = 0.$$

The obtained contradiction shows that there is no Nash equilibrium in pure strategies.

Proof. (necessity)

Let us assume that there is no Nash equilibrium in pure strategies and there exists a column k such that $a_{ik} \leq 0$ for each $i = 1, 2, \ldots, n$. This implies that $\sum_{i=1}^{n} a_{ik} x_i \leq 0$ for each mixed strategy $x = (x_1, x_2, \ldots, x_n)$, i.e. $\sum_{i=1}^{n} -a_{ki} x_i \leq 0$, and hence $\sum_{j=1}^{n} a_{kj} x_j \geq 0$. So, we obtain that $P(x, k) \leq 0$ and $P(k, x) \geq 0$. This means that (k, k) is a Nash equilibrium in pure strategies.

Symmetric games

Let us consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

and let us define the following $m+n+1 \times m+n+1$ skew-symmetric matrix:

$$B = \begin{pmatrix} O_{m\times m} & A_{m\times n} & -I_{m\times 1} \\ -A^T_{n\times m} & O_{n\times n} & I_{n\times 1} \\ I^T_{1\times m} & -I^T_{1\times n} & O_{1\times 1} \end{pmatrix}$$

where $O_{k\times k}$ is a $k \times k$ zero matrix and $I^T_{1\times m}$ is a vector-raw with m components equal to 1.

Two-person zero-sum games

Theorem.

Let the elements of the matrix A are positive. Let $((\bar{x}, \bar{y}, \lambda), (\bar{x}, \bar{y}, \lambda))$ be a Nash equilibrium of the game determined of the matrix B. Then
(i) $\sum_{i=1}^{m} \bar{x}_i = \sum_{j=1}^{n} \bar{y}_j = \mu > 0$;
(ii) the couple $(\bar{x}/\mu, \bar{y}/\mu)$ is a Nash equilibrium of the game determined by the matrix A;
(iii) the value of the game determined by the matrix A is λ/μ.

Proof.

We have that

$$\sum_{i=1}^{m} \bar{x}_i + \sum_{j=1}^{n} \bar{y}_j + \lambda = 1,$$

with $\bar{x}_i \geq 0$, $i = 1, \ldots, m$, $\bar{y}_j \geq 0$, $j = 1, \ldots, n$, and $\lambda \geq 0$.

Proof (continuation).

Since $((\bar{x}, \bar{y}, \lambda), (\bar{x}, \bar{y}, \lambda))$ is a Nash equilibrium of the game determined of the matrix B, the following inequalities hold true:

$$\sum_{j=1}^{n} a_{ij}\bar{y}_j - \lambda \leq 0, \;\; i = 1, \ldots, m, \tag{3}$$

$$-\sum_{i=1}^{m} a_{ij}\bar{x}_i + \lambda \leq 0, \;\; j = 1, \ldots, n, \tag{4}$$

$$\sum_{i=1}^{m} \bar{x}_i - \sum_{j=1}^{n} \bar{y}_j \leq 0. \tag{5}$$

Let us assume that $\lambda = 0$. This contradicts to (3).
Let us assume that $\lambda = 1$. Then $\bar{x} = 0$ and $\bar{y} = 0$. But this contradicts to (4).
Hence $\lambda \in (0, 1)$. Then

$$\sum_{i=1}^{m} \bar{x}_i + \sum_{j=1}^{n} \bar{y}_j = 1 - \lambda =: 2\mu$$

and

$$\sum_{i=1}^{m} \bar{x}_i - \sum_{j=1}^{n} \bar{y}_j = 0, \text{ i.e. } \sum_{i=1}^{m} \bar{x}_i = \sum_{j=1}^{n} \bar{y}_j = \mu.$$

Then

$$\sum_{i=1}^{m} \frac{\bar{x}_i}{\mu} = \sum_{j=1}^{n} \frac{\bar{y}_j}{\mu} = 1$$

$$\sum_{j=1}^{n} a_{ij} \frac{\bar{y}_j}{\mu} \leq \frac{\lambda}{\mu}, \ \frac{\bar{x}_i}{\mu} \geq 0, \ i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} a_{ij} \frac{\bar{x}_i}{\mu} \geq \frac{\lambda}{\mu}, \ \frac{\bar{y}_j}{\mu} \geq 0, \ j = 1, \ldots, n.$$

Applying Lemma 6, we obtain that the couple $(\bar{x}/\mu, \bar{y}/\mu)$ is a Nash equilibrium of the game determined by the matrix A.

Two-person zero-sum games

Theorem.

Let the elements of the matrix A are positive. Let the couple $(\bar{x}, \bar{y})$ be a Nash equilibrium of the game determined by the matrix A with value v. Then

$$\left(\frac{\bar{x}}{2+v}, \frac{\bar{y}}{2+v}, \frac{v}{2+v} \right).$$

is a Nash equilibrium of the game determined of the matrix B.

Proof.

Since $(\bar{x}, \bar{y})$ is a Nash equilibrium of the game determined by the matrix A, we have that

$$\sum_{i=1}^{m} \bar{x}_i = \sum_{j=1}^{n} \bar{y}_j = 1$$

Two-person zero-sum games

Proof (continuation).

$$\sum_{j=1}^{n} a_{ij}\bar{y}_j \leq v,\ \ \bar{x}_i \geq 0,\ \ i = 1, \ldots, m,$$

$$\sum_{i=1}^{m} a_{ij}\bar{x}_i \geq v,\ \ \bar{y}_j \geq 0,\ \ j = 1, \ldots, n.$$

Proof (continuation).

These inequalities imply that

$$\sum_{j=1}^{n} a_{ij} \frac{\bar{y}_j}{v+2} - \frac{v}{v+2} \leq 0, \ \ \bar{x}_i \geq 0, \ \ i = 1, \ldots, m,$$

$$-\sum_{i=1}^{m} a_{ij} \frac{\bar{x}_i}{v+2} + \frac{v}{v+2} \geq 0, \ \ \bar{y}_j \geq 0, \ \ j = 1, \ldots, n.$$

Applying Lemma 6, we obtain that the couple

$$\left(\frac{\bar{x}_i}{v+2}, \frac{\bar{y}_j}{v+2}, \frac{v}{v+2}\right)$$

is a Nash equilibrium of the game determined by the matrix B.