(3) Calcolare l'integrale (curvilineo) di

$$f(x,y)=\frac{xy}{\sqrt{4+x^2}}$$

lungo la curva γ il cui sostegno è il bordo ∂E di

$$E=\left\{(x,y): x\geq 0,\ x^2+y^2\geq 1,\ 0\leq y\leq 1-\frac{x^2}{4}\right\}$$

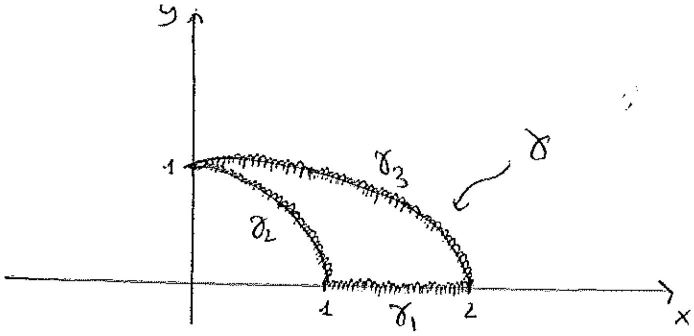
Iniziamo disegnandoci il dominio

Consideriamo $y=1-\frac{x^2}{4}$

Si ha

$$y(0)=1,\quad y'=-\frac{x}{2}\ \Rightarrow\ V=(0,1),\quad 1-\frac{x^2}{4}=0\ \Rightarrow\ x=\pm 2$$

Pertanto abbiamo una parabola di vertice $V=(0,1)$ che interseca l'asse x nei punti $x=2$ e $x=-2$.



Scriviamo l'espressione delle curve γ_1, γ_2 e γ_3 in forma parametrica.
Si ha

$$\gamma_1(t)=\begin{pmatrix}2-t\\0\end{pmatrix}\quad \text{con } t\in[0,1]\ \Rightarrow\ \dot{\gamma}_1(t)=\begin{pmatrix}-1\\0\end{pmatrix}$$

$$\gamma_2(t)=\begin{pmatrix}\cos\vartheta\\ \sin\vartheta\end{pmatrix}\quad \text{con } \vartheta\in[0,\pi/2]\ \Rightarrow\ \dot{\gamma}_2(t)=\begin{pmatrix}-\sin\vartheta\\ \cos\vartheta\end{pmatrix}$$

$$\gamma_3(t)=\begin{pmatrix}x\\1-\frac{x^2}{4}\end{pmatrix}\quad \text{con } x\in[0,2]\ \Rightarrow\ \dot{\gamma}_3(t)=\begin{pmatrix}1\\-\frac{x}{2}\end{pmatrix}$$

f.q.

Pertanto

$$\int_\gamma f(x,y)\,d\ell=\int_{\gamma_1\cup\gamma_2\cup\gamma_3} f(x,y)\,d\ell=\int_0^1 f(\gamma_1(t))\cdot|\dot{\gamma}_1(t)|\,dt+\int_0^{\pi/2} f(\gamma_2(\vartheta))\cdot|\dot{\gamma}_2(\vartheta)|\,d\vartheta\ +$$

$$+\int_0^2 f(\gamma_3(t))\cdot|\dot{\gamma}_3(x)|\,dx$$

Calcoliamo separatamente i tre integrali:

1) $\displaystyle\int_0^1 f(\gamma_1(t))\cdot|\dot{\gamma}_1(t)|\,dt = 0$ poiché $f(\gamma_1(t)) = 0$

$z = \cos\theta \quad dz = -\sin\theta\, d\theta$

$w = 4+z^2$

$d(4+z^2) = 2z\,dz$

2) $\displaystyle\int_0^{\pi/2} f(\gamma_2(\theta))\cdot|\dot{\gamma}_2(\theta)|\,d\theta = \int_0^{\pi/2} \frac{\cos\theta\sin\theta}{\sqrt{4+\cos^2\theta}}\,d\theta = -\int_1^0 \frac{z\,dz}{\sqrt{4+z^2}} = \int_0^1 \frac{z}{\sqrt{4+z^2}}\,dz = \frac{1}{2}\int_4^5 \frac{dw}{\sqrt{w}} =$

$$= \frac{1}{2}\Big[2\sqrt{w}\Big]_4^5 = \sqrt{5}-2$$

3) $\displaystyle\int_0^2 f(\gamma_3(x))\cdot|\dot{\gamma}_3(x)|\,dx = \int_0^2 \frac{x\left(1-\frac{x^2}{4}\right)}{\sqrt{4+x^2}}\cdot\sqrt{1+\frac{x^2}{4}}\,dx = \frac{1}{2}\int_0^2\left(x-\frac{x^3}{4}\right)dx =$

$$= \frac{1}{2}\left[\frac{x^2}{2}-\frac{x^4}{16}\right]_0^2 = \frac{1}{2}$$

Quindi

$$\int_\gamma f(x,y)\,d\ell = \sqrt{5}-2+\frac{1}{2} = \sqrt{5}-\frac{3}{2}$$