Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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Lecture IX

1 Submersions

Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^{k}$ (or $\mathcal{U} \subset \mathbb{R}^{n+k}$ open) $\quad k \geq 1$

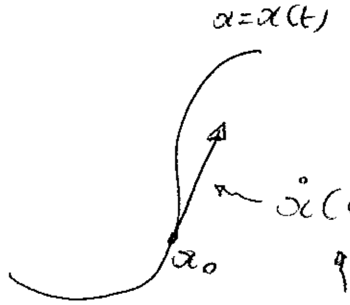
be a smooth function.

$$f : \underbrace{(x^1 \ldots x^{n+k})}_{x} \longmapsto \underbrace{(y^1 \ldots y^k)}_{y} \qquad y^i = f^i(x)$$

$$y = f(x)$$

Let $\begin{cases} x = x(t) \\ x(0) = x_0 \in \mathcal{U} \end{cases}$ $\quad t \in I$, interval containing 0

be a smooth curve in $\mathbb{R}^{n+k}$ $\qquad$ 0 $\quad$ I



$\dot{x}(0) = h \in \mathbb{R}^{n+k}$

velocity of the curve at x_0 (corresponding to $t=0$)

Let $F(t) := f(x(t)) = (f \circ x)(t)$

Then

$$\left.\frac{d}{dt} F(t)\right|_{t=0} = (f_*)_{x_0}(h)$$

$(f_*)_{x_0} \equiv df|_{x_0}$ alternative notation for the differential

f_* is also called push-forward

Obviously

$$(f_*)_{x_0} \, h = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \frac{\partial f^1}{\partial x^2} & \cdots & \frac{\partial f^1}{\partial x^{n+k}} \\ \vdots & & & \\ \frac{\partial f^k}{\partial x^1} & \frac{\partial f^k}{\partial x^2} & \cdots & \frac{\partial f^k}{\partial x^{n+k}} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{n+k} \end{pmatrix}$$

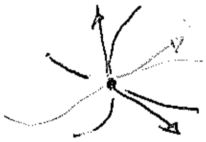
$h \in \mathbb{R}^{n+k}$

differential $\equiv$ push-forward

Jacobian matrix

$k \times (n+k)$ - matrix

can be viewed as velocity vectors of curves emanating from x_0



Definition: f is a submersion at x_0 (or, equivalently, f is submersive at x_0) if $f_*|_{x_0}$ is surjective

★ This yields yet another interpretation of tangent vectors, also crucial for the sequel

This is tantamount to require that

$$\underset{\text{rank}}{r(f_*|_{x_0})} = \dim \operatorname{Im}(f_*|_{x_0}) = \left(\begin{smallmatrix}\text{rank of the}\\ \text{Jacobian matrix}\end{smallmatrix}\right) = k$$

This entails, by the N + R - theorem, that

$$\underset{\text{nullity}}{\nu(f_*|_{x_0})} = \dim \ker(f_*|_{x_0}) = n + k - k = n$$

We shall need this fact very soon.

Definition. A subset $M \subset \mathbb{R}^{n+k}$ is said to be a (smooth) n-dimensional submanifold of $\mathbb{R}^{n+k}$

sottovarietà (liscia)

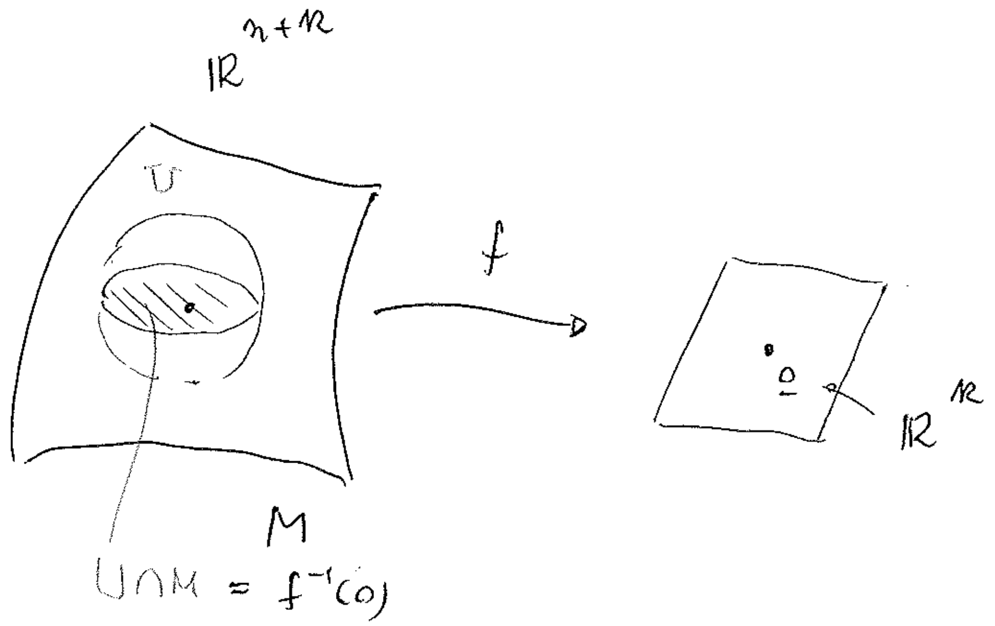
k: codimension of M
codimensione

if $\forall x \in M$, $\exists$ $U \ni x$ in $\mathbb{R}^{n+k}$
neighbourhood

and a submersion $f : U \to \mathbb{R}^k$ (smooth...)

such that $U \cap M = f^{-1}(0)$

level set of f
(pertaining to $0 \in \mathbb{R}^k$)



Given $x_0 \in f^{-1}(0)$, $\ker f_*|_{x_0} \leq \mathbb{R}^{n+k}$
subspace

is called the tangent space to M at x_0

Let us justify the last assertion

A moment's reflection shows that $\mathrm{Ker}\, f_*|_{x_0}$ (of dimension n) consists of the velocity vectors of curves issuing from x_0 and entirely lying in $U \cap M = f^{-1}(0)$ for sufficiently small t: Indeed, let x_0, $f(x_0) = 0$.

Let $x = x(t)$ $\quad x(0) = x_0$ $\quad t \in I \quad I \ni 0$
$(\dot{x} = \dot{x}(t) \quad \dot{x}(0) = \xi \in \mathbb{R}^{n+k})$

be a curve lying in $f^{-1}(0)$: $F(t) := f(x(t)) \equiv 0 \quad \forall\, t \in I$

Then, differentiation with respect to t yields

$$\frac{\partial f}{\partial x^i}\,\dot{x}^i \equiv 0$$

, and, in particular, at $t = 0$

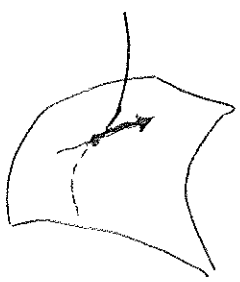
$$(f_*)_{x_0}\,\xi = 0$$

$$\begin{array}{cc} & n+k \\ k & \boxed{\; f_*|_{x_0} \;} \end{array} \; \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{\xi} = \underbrace{\begin{bmatrix} \\ \\ \end{bmatrix}}_{0} \in \mathbb{R}^k$$

Conversely, as a consequence of Dini's theory, any vector in $\mathrm{Ker}\, f_*|_{x_0}$ can be viewed as the velocity vector of a curve stemming from x_0 (try to figure out this...)†

We shall discuss several examples.

† Clearly, a vector in $\mathrm{Ker}\, f_*|_{x_0}$ can be the velocity vector of curves leaving M. The point is that curves staying in M can be produced.

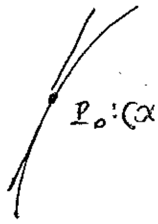


Examples

1. <u>Curves in $\mathbb{R}^2$</u> (defined implicitly) $\quad f \in \mathcal{C}^\infty(\mathbb{R}^2)$, $f: \mathbb{R}^2 \to \mathbb{R}$

$\mathcal{C}: \quad f(x,y) = 0$

(level set of 0)



($\diamond$) $\quad \dfrac{\partial f}{\partial y}(P_0) \neq 0$

$P_0 : (x_0, y_0)$

$\Rightarrow$ locally $\quad y = y(x) \quad y(x_0) = y_0$

$f(x, y(x)) \equiv 0$

($\diamond$) ensures that f is submersive at P_0

$$f_*\big|_{P_0} = (f_x^0, f_y^0) \quad (\Rightarrow r(f_*|_{P_0}) = 1 = \max)$$
$\qquad\qquad\qquad\qquad \neq 0$

The tangent space to $\mathcal{C}$ at P_0 is precisely the (direction of the) tangent line to $\mathcal{C}$ at P_0; in fact.

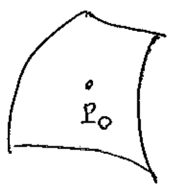
(caveat...)

$$\Big[\, f_x^0 \underbrace{(x - x_0)}_{\xi_1} + f_y^0 \underbrace{(y - y_0)}_{\xi_2} = 0 \,\Big] \Leftrightarrow \boxed{\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \operatorname{Ker} f_*\big|_{P_0}}$$

equation of the tangent to $\mathcal{C}$ at P_0

2. <u>Surfaces in $\mathbb{R}^3$</u> (defined implicitly)

$\Sigma: \quad f(x,y,z) = 0 \qquad f: \mathbb{R}^3 \to \mathbb{R} \quad$ (smooth)



tangent plane to Σ at P_0

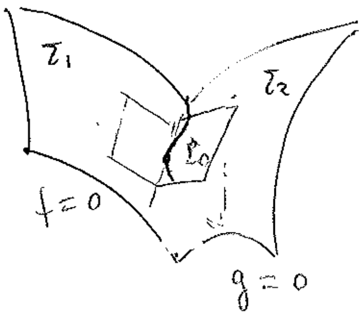
$$\Big[\, f_x^0 \underbrace{(x - x_0)}_{\xi_1} + f_y^0 \underbrace{(y - y_0)}_{\xi_2} + f_z^0 \underbrace{(z - z_0)}_{\xi_3} = 0 \,\Big]$$
$$\Updownarrow$$
$$\boxed{\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} \in \operatorname{Ker} f_*\big|_{P_0}}$$

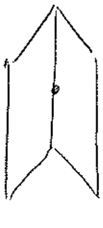
$\dfrac{\partial f}{\partial z}(P_0) \neq 0$ ensures f to be a submersion

$\Rightarrow z = z(x,y)$

$$f_*\big|_{P_0} = (f_x^0, f_y^0, f_z^0)$$

3. Curves in $\mathbb{R}^3$





$$\mathcal{C} : \begin{cases} f(x,y,z) = 0 \\ g(x,y,z) = 0 \end{cases} \qquad F : \begin{array}{c} \mathbb{R}^3 \\ \Downarrow \\ (x,y,z) \\ \downarrow \\ (f(x,y,z), g(x,y,z)) \\ \Uparrow \\ \mathbb{R}^2 \end{array}$$

$$\mathcal{C} = F^{-1}(0)$$

tangent to $\mathcal{C}$ in P_0

tangent plane to Σ_1 in P_0: $\quad f_x^0 \overbrace{(x-x_0)}^{\xi_1} + f_y^0 \overbrace{(y-y_0)}^{\xi_2} + f_z^0 \overbrace{(z-z_0)}^{\xi_3} = 0$

... to Σ_2 ...: $\quad g_x^0(x-x_0) + g_y^0(y-y_0) + g_z^0(z-z_0) = 0$

$$\underbrace{\begin{pmatrix} f_x^0 & f_y^0 & f_z^0 \\ g_x^0 & g_y^0 & g_z^0 \end{pmatrix}}_{F_*|_{P_0}} \underbrace{\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}}_{= \xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \xi \in \operatorname{Ker} F_*|_{P_0}$$

4. $$S^n = \{ x \in \mathbb{R}^{n+1} : f(x) = x_0^2 + x_1^2 + \dots x_n^2 - 1 = 0 \}$$

(the n-dim. sphere in $\mathbb{R}^{n+1}$) example of a hypersurface

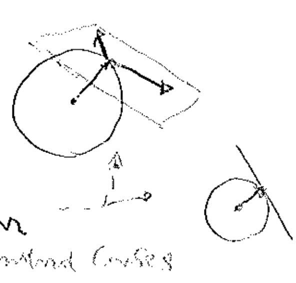
$$f_*|_x = (2x_0, 2x_1, \dots, 2x_n) \in (\mathbb{R}^{n+1})^*.$$

$$f_*|_x \underset{\begin{pmatrix} h_0 \\ \vdots \\ h_n \end{pmatrix}}{\underset{\parallel}{h}} = 2 \sum_{i=1}^{n} x_i h_i$$

f_* is clearly surjective for all $x \in S^n$ (since $(x_0, \dots x_n) \neq (0, \dots, 0)$ $\forall x \in S^n$, in view of $\sum_{i=0}^{n} x_i^2 = 1$)

* tangent space (at x_0) $\qquad \sum_{i=0}^{n} x_i^0 (x_i - x_i^0) = 0$

x_0 and $x - x_0$ must be perpendicular



cf the standard cases

5. $H^n_c = \{ x \in \mathbb{R}^{n+1} \,/\, g_c(x) = x_0^2 - x_1^2 - x_2^2 - \dots - x_n^2 - c = 0 \}$

hyperboloid

For $c \neq 0$, H^n_c is a submanifold (actually, a hypersurface) of $\mathbb{R}^{n+1}$

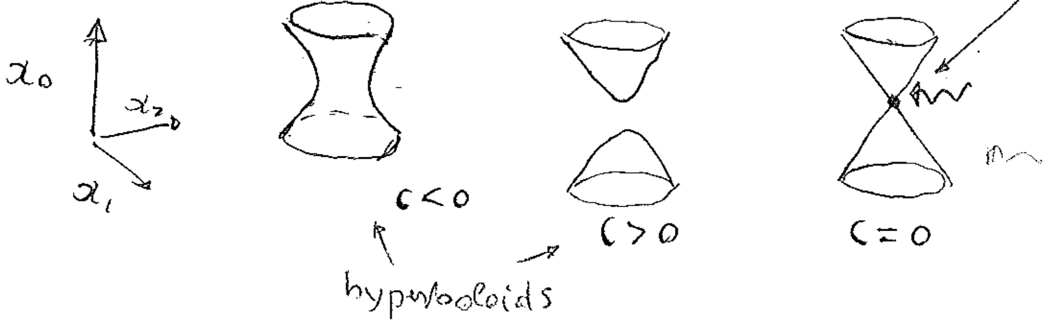
$$(g_c)_*\big|_x = (2x_0, -2x_1, \dots -2x_n)$$

surjective $\forall\; c \neq 0$, $\forall x \in H^n_c$

If $c = 0$ $\quad (g_0)_*\big|_0 = 0$ (hence not surjective)

$H^n_0 \setminus \{0\}$ is a submanifold (a "cone" without apex)

The tangent plane cannot be defined!



notice that this is not homeomorphic to a disc.

6. n-dimensional torus $\mathbb{T}^n = \{ z = (z_1 \dots z_n) \,/\, |z_i|^2 = 1 \}$

$$= \{ x = (x_1 \dots x_{2n}) \,/\, f(x) \equiv (x_1^2 + x_2^2 - 1, \dots, x_{2n-1}^2 + x_{2n}^2 - 1) = (0, \dots, 0) \}$$

$\mathbb{T}^n = S^1 \times S^1 \times \dots S^1$

It is a submanifold of $\mathbb{R}^{2n}$ $(\cong \mathbb{C}^n)$

7. This example is quite important and instructive

$$SO(n) = \{ A \in \underset{\substack{n\times n \\ \text{matrices}}}{M_n(\mathbb{R})} \,/\, \underbrace{A^T A = A A^T = I_n}_{\substack{\text{This condition} \\ \text{defines } O(n) \\ \text{orthogonal group}}} ,\ \det A = 1 \}$$

special orthogonal group

(isometries of the euclidean vector space $(\mathbb{R}^n, \langle , \rangle_{\text{standard}})$ with determinant $= +1$

Recall that if $A \in O(n)$, then $\det A = \pm 1$ (the converse is obviously false)

$SO(n)$ is a submanifold of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$, having dimension $\dfrac{n(n-1)}{2}$

* Let us realize it as a level set $f^{-1}(0)$

Let $f : GL_n^+(\mathbb{R}) \longrightarrow \mathrm{Sym}_n$

non singular matrices with positive determinant

symmetric matrices. $B^T = B$

$$A \longmapsto f(A) := A^T A - I_n \quad (\text{clear})$$

Then $f^{-1}(\underset{\mathrm{Sym}_n}{\underset{\cap}{0}}) = SO(n)$

Let us check that f is a submersion; its differential reads

$$\boxed{f^*\big|_{\underset{SO(n)}{\underset{\cap}{A}}} \underset{M_n}{\underset{\cap}{H}} = A^T H + H^T A}$$

Indeed, let $A = A(t)$, $t \in I$ (I an interval containing 0) be a smooth curve in $Gl_n^+(\mathbb{R})$ such that $A(0) = A$, $\mathring{A}(0) = H$ (for instance, $A(t) = A + tH$, for t small enough)

We have to compute:

$$\frac{d}{dt}\left[A(t)^T A(t) - I_n\right]\Big|_{t=0}, \text{ getting;}$$

successively,

$$\underbrace{\mathring{A}^T(0)}_{\substack{\| \\ \mathring{A}(0)^T \\ \| \\ H^T}} \underbrace{A(0)}_{\substack{\| \\ A}} + \underbrace{A(0)^T}_{\substack{\| \\ A^T}} \underbrace{\mathring{A}(0)}_{\substack{\| \\ H}} = H^T A + A^T H$$

derivation commutes with transposition

Now let $S \in Sym_n$ <u>any</u> real symmetric matrix.

Set $H := \frac{AS}{2}$. Then

$$\left(\frac{AS}{2}\right)^T A + A^T \frac{AS}{2} = \frac{1}{2} \underbrace{S^T}_{\substack{\| \\ S}} \underbrace{A^T A}_{\substack{\| \\ I_n}} + \frac{1}{2} \underbrace{A^T A}_{I_n} S = S,$$

hence $f_*|_A$ is surjective.

* The tangent space to $SO(n)$ at A – notation: $T_A SO(n)$ –

$$\operatorname{Ker} f_*|_A = \{ H \in M_n(\mathbb{R}) \,/\, H^T A + A^T H = 0 \}$$

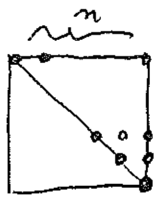
In particular, $T_{I_n} SO(n) = \{ H \,/\, H^T + H = 0 \}$

that is, the antisymmetric / skewsymmetric $n \times n$ – matrices

[This is the Lie algebra $so(n)$ of the Lie group $SO(n)$]

Further remarks

$$\dim \mathrm{Sym}_n = \frac{n(n+1)}{2} = \operatorname{codim} \mathfrak{so}(n)$$



Since it equals $1+2+\dots n$

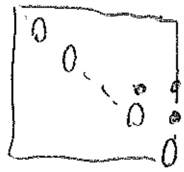
$$\dim \mathfrak{so}(n) = n^2 - \frac{n(n+1)}{2}$$

$$= \frac{2n^2 - n^2 - n}{2} = \frac{n^2-n}{2} = \frac{n(n-1)}{2}$$

This is also clear from the fact that $\dim \mathfrak{so}(n) = \frac{n(n-1)}{2}$

⚠

$$1+2+\dots+(n-1) = \frac{n(n-1)}{2}$$



Let $A \in M_n(\mathbb{R})$. Then $A = \underbrace{\frac{A+A^T}{2}}_{\in\, \mathrm{Sym}_n} + \underbrace{\frac{A-A^T}{2}}_{\in\, \mathfrak{so}(n)}$

and this expression is unique; that is

$$M_n = \mathrm{Sym}_n \oplus \mathfrak{so}(n)$$

direct sum

Actually, we have an orthogonal direct sum upon setting

$$\langle A, B\rangle := \underset{\text{trace}}{\mathrm{Tr}}(A^T B)$$

Frobenius, or Hilbert-Schmidt inner product

check that this defines an inner product, and that $\langle A,B\rangle = 0$ if $A \in \mathrm{Sym}_n$, $B \in \mathfrak{so}(n)$

recall that for any $n\times n$ matrix $A = (a_{ij})$, $\mathrm{Tr}(A) := \sum_{i=1}^{n} a_{ii}$, and one has $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, whence from one proves the cyclical property $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$

and $\mathrm{Tr}(S^{-1}AS) = \mathrm{Tr}\, A$ (similarity invariance)